Unit-1

QUADRATIC EQUATIONS

In this unit, students will learn how to

- *define quadratic equation.*
- *solve a quadratic equation in one variable by factorization.*
- *solve a quadratic equation in one variable by completing square.*
- *derive quadratic formula by using method of completing square.*
- *solve a quadratic equation by using quadratic formula.*
- *solve the equations of the type* $ax^4 + bx^2 + c = 0$ *by reducing it to the quadratic form.*
- *solve the equations of the type* $a\,p(x) + \dfrac{b}{p(x)} = c$.
- *solve reciprocal equations of the type* $a\left(x^2 + \dfrac{1}{x^2}\right) + b\left(x + \dfrac{1}{x}\right) + c = 0$.
- *solve exponential equations involving variables in exponents.*
- *solve equations of the type* $(x + a)(x + b)(x + c)(x + d) = k$ *where* $a + b = c + d$.
- *solve radical equations of the types*
  (i) $\sqrt{ax + b} = cx + d$,
  (ii) $\sqrt{x + a} + \sqrt{x + b} = \sqrt{x + c}$,
  (iii) $\sqrt{x^2 + px + m} + \sqrt{x^2 + px + n} = q$.

1.1. Quadratic Equation

An equation, which contains the square of the unknown (variable) quantity, but no higher power, is called a **quadratic equation** or an equation of the **second degree**.

A second degree equation in one variable x of the form

$ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, c are real numbers,

is called the **general or standard form** of a quadratic equation.

Here a is the co-efficient of x^2, b is the co-efficient of x and constant term is c.

The equations $x^2 - 7x + 6 = 0$ and $3x^2 + 4x = 5$ are the examples of quadratic equations.

Remember that: If $a = 0$ in $ax^2 + bx + c = 0$, then it reduces to a linear equation $bx + c = 0$.

$x^2 - 7x + 6 = 0$ is in standard form but

$3x^2 + 4x = 5$ is not in standard form.

If $b = 0$ in a quadratic equation $ax^2 + bx + c = 0$, then it is called a **pure quadratic** equation. For example $x^2 - 16 = 0$ and $4x^2 = 7$ are the pure quadratic equations.

Activity: Write two pure quadratic equations.

1.2 Solution of quadratic equations

To find solution set of a quadratic equation, following methods are used:

(i) factorization (ii) completing square

1.2(i) Solution by factorization:

In this method, write the quadratic equation in the standard form as

$$ax^2 + bx + c = 0 \qquad \text{(i)}$$

If two numbers r and s can be found for equation (i) such that $r + s = b$ and $rs = ac$, then $ax^2 + bx + c$ can be factorized into two linear factors.

The procedure is explained in the following examples.

Example 1: Solve the quadratic equation $3x^2 - 6x = x + 20$ by factorization.

Solution: $3x^2 - 6x = x + 20$ (i)

The standard form of (i) is $3x^2 - 7x - 20 = 0$ (ii)

Here $a = 3$, $b = -7$, $c = -20$ and $ac = 3 \times -20 = -60$

As $-12 + 5 = -7$ and $-12 \times 5 = -60$, so

the equation (ii) can be written as

$3x^2 - 12x + 5x - 20 = 0$

or $3x(x - 4) + 5(x - 4) = 0$

$\Rightarrow$ $(x - 4)(3x + 5) = 0$

Either $x - 4 = 0$ or $3x + 5 = 0$, that is,

Activity: Factorize $x^2 - x - 2 = 0$.

$x = 4$ or $3x = -5 \Rightarrow x = -\frac{5}{3}$

$\therefore$ $x = -\frac{5}{3}$, 4 are the solutions of the given equation.

Thus, the solution set is $\left\{-\frac{5}{3}, 4\right\}$.

Example 2: Solve $5x^2 = 30x$ by factorization.

Solution: $5x^2 = 30x$

$5x^2 - 30x = 0$ which is factorized as

$5x(x - 6) = 0$

> **Remember that:** Cancelling of x on both sides of $5x^2 = 30x$ means the loss of one root *i.e.*, $x = 0$

Either $5x = 0$ or $x - 6 = 0 \Rightarrow x = 0$ or $x = 6$

$\therefore$ $x = 0, 6$ are the roots of the given equation.

Thus, the solution set is $\{0, 6\}$.

1.2(ii) Solution by completing square:

To solve a quadratic equation by the method of completing square is illustrated through the following examples.

Example 1: Solve the equation $x^2 - 3x - 4 = 0$ by completing square.

Solution: $x^2 - 3x - 4 = 0$ (i)

Shifting constant term -4 to the right, we have

$x^2 - 3x = 4$ (ii)

Adding the square of $\frac{1}{2} \times$ coefficient of x, that is,

$\left(-\frac{3}{2}\right)^2$ on both sides of equation (ii), we get

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 4 + \left(-\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = 4 + \frac{9}{4} = \frac{16 + 9}{4}$$

or $$\left(x - \frac{3}{2}\right)^2 = \frac{25}{4}$$

Taking square root of both sides of the above equation, we have

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{\frac{25}{4}}$$

$$\Rightarrow \quad x - \frac{3}{2} = \pm\frac{5}{2} \text{ or } x = \frac{3}{2} \pm \frac{5}{2}$$

Either $x = \frac{3}{2} + \frac{5}{2} = \frac{3+5}{2} = \frac{8}{2} = 4$ or $x = \frac{3}{2} - \frac{5}{2} = \frac{3-5}{2} = \frac{-2}{2} = -1$

$\therefore$ 4 and -1 are the roots of the given equation.

Thus, the solution set is $\{-1, 4\}$.

Example 2: Solve the equation $2x^2 - 5x - 3 = 0$ by completing square.

Solution: $2x^2 - 5x - 3 = 0$

Dividing each term by 2, to make coefficient of x^2 equal to 1.

$$x^2 - \frac{5}{2}x - \frac{3}{2} = 0$$

Remember that: For our convenience, we make the co-efficient of x^2 equal to 1 in the method of completing square.

Shifting constant term $-\frac{3}{2}$ to the right

$$x^2 - \frac{5}{2}x = \frac{3}{2} \qquad \text{(i)}$$

Multiply co-efficient of x with $\frac{1}{2}$ *i.e.*, $\frac{1}{2}\left(-\frac{5}{2}\right) = -\frac{5}{4}$

Now adding $\left(-\frac{5}{4}\right)^2$ on both sides of the equation (i), we have

$$x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 = \frac{3}{2} + \left(-\frac{5}{4}\right)^2, \text{ that is,}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16} = \frac{24+25}{16}$$

or $$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16}$$

Taking square root of both sides of the above equation, we have

$$\sqrt{\left(x - \frac{5}{4}\right)^2} = \pm\sqrt{\frac{49}{16}}$$

$$\Rightarrow \quad x - \frac{5}{4} = \pm\frac{7}{4}$$

Either $x - \frac{5}{4} = \frac{7}{4}$ or $x - \frac{5}{4} = -\frac{7}{4}$, that is,

$$x = \frac{7}{4} + \frac{5}{4} \quad \text{or} \quad x = -\frac{7}{4} + \frac{5}{4}$$

$$= \frac{7+5}{4} = \frac{12}{4} = 3 \quad \text{or} \quad = \frac{-7+5}{4} = \frac{-2}{4} = \frac{-1}{2}$$

$\therefore$ $x = -\frac{1}{2}, 3$ are the roots of the given equation.

Thus, the solution set is $\left\{-\frac{1}{2}, 3\right\}$.

EXERCISE 1.1

1. Write the following quadratic equations in the standard form and point out pure quadratic equations.

(i). $(x + 7)(x - 3) = -7$ (ii) $\frac{x^2 + 4}{3} - \frac{x}{7} = 1$

(iii) $\frac{x}{x+1} + \frac{x+1}{x} = 6$ (iv) $\frac{x+4}{x-2} - \frac{x-2}{x} + 4 = 0$

(v) $\frac{x+3}{x+4} - \frac{x-5}{x} = 1$ (vi) $\frac{x+1}{x+2} + \frac{x+2}{x+3} = \frac{25}{12}$

2. Solve by factorization:

(i) $x^2 - x - 20 = 0$ (ii) $3y^2 = y(y - 5)$

(iii) $4 - 32x = 17x^2$ (iv) $x^2 - 11x = 152$

(v) $\frac{x+1}{x} + \frac{x}{x+1} = \frac{25}{12}$ (vi) $\frac{2}{x-9} = \frac{1}{x-3} - \frac{1}{x-4}$

3. Solve the following equations by completing square:

(i) $7x^2 + 2x - 1 = 0$ (ii) $ax^2 + 4x - a = 0\ ,\ a \neq 0$

(iii) $11x^2 - 34x + 3 = 0$ (iv) $lx^2 + mx + n = 0,\ l \neq 0$

(v) $3x^2 + 7x = 0$ (vi) $x^2 - 2x - 195 = 0$

(vii) $-x^2 + \frac{15}{2} = \frac{7}{2}x$ (viii) $x^2 + 17x + \frac{33}{4} = 0$

(ix) $4 - \frac{8}{3x+1} = \frac{3x^2+5}{3x+1}$ (x) $7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$

1.3 Quadratic Formula:

1.3. (i) Derivation of quadratic formula by using completing square method.

The quadratic equation in standard form is

$$ax^2 + bx + c = 0\quad ,\quad a \neq 0$$

Dividing each term of the equation by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Shifting constant term $\frac{c}{a}$ to the right, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ on both sides, we obtain

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a}$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square root of both sides, we get

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

or $$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow$ $$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$**, $\boldsymbol{a \neq 0}$** is known as **"quadratic formula"**.

1.3 (ii) Use of quadratic formula:

The quadratic formula is a useful tool for solving all those equations which can or can not be factorized. The method to solve the quadratic equation by using quadratic formula is illustrated through the following examples.

Example 1: Solve the quadratic equation $2 + 9x = 5x^2$ by using quadratic formula.

Solution: $2 + 9x = 5x^2$

The given equation in standard form can be written as

$$5x^2 - 9x - 2 = 0$$

Comparing with the standard quadratic equation $ax^2 + bx + c = 0$, we observe that

$$a = 5\ ,\ b = -9\ ,\ c = -2$$

Putting the values of a, b and c in quadratic formula

Activity: Using quadratic formula, write the solution set of $x^2 + x - 2 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$, we have

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(5)(-2)}}{2(5)}$$

or $$x = \frac{9 \pm \sqrt{81 + 40}}{10} = \frac{9 \pm \sqrt{121}}{10} = \frac{9 \pm 11}{10}$$

Either $x = \frac{9 + 11}{10}$ or $x = \frac{9 - 11}{10}$, that is,

$x = \frac{20}{10} = 2$ or $x = \frac{-2}{10} = -\frac{1}{5}$

$\therefore$ $2, -\frac{1}{5}$ are the roots of the given equation.

Thus, the solution set is $\left\{-\frac{1}{5}, 2\right\}$.

Example 2: Using quadratic formula, solve the equation $\frac{2x+1}{x+2} - \frac{x-2}{x+4} = 0$.

Solution: $\frac{2x+1}{x+2} - \frac{x-2}{x+4} = 0$

Simplifying and writing in the standard form

$(2x+1)(x+4) - (x-2)(x+2) = 0$

$2x^2 + 8x + x + 4 - (x^2 - 4) = 0$

$2x^2 + 9x + 4 - x^2 + 4 = 0$

or $x^2 + 9x + 8 = 0$

Here $a = 1$, $b = 9$, $c = 8$

Using quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we have

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4 \times 1 \times 8}}{2 \times 1}$$

$$= \frac{-9 \pm \sqrt{81 - 32}}{2} = \frac{-9 \pm \sqrt{49}}{2} = \frac{-9 \pm 7}{2}$$

$\Rightarrow$ $x = \frac{-9+7}{2} = \frac{-2}{2} = -1$

or $x = \frac{-9-7}{2} = \frac{-16}{2} = -8$

$\therefore$ $-1, -8$ are the roots of the given equation. Thus, the solution set is $\{-8, -1\}$.

EXERCISE 1.2

1. Solve the following equations using quadratic formula:

(i) $2 - x^2 = 7x$

(ii) $5x^2 + 8x + 1 = 0$

(iii) $\sqrt{3}\,x^2 + x = 4\sqrt{3}$

(iv) $4x^2 - 14 = 3x$

(v) $6x^2 - 3 - 7x = 0$

(vi) $3x^2 + 8x + 2 = 0$

(vii) $\frac{3}{x-6} - \frac{4}{x-5} = 1$

(viii) $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}$

(ix) $\frac{a}{x-b} + \frac{b}{x-a} = 2$

(x) $-(l+m) - lx^2 + (2l+m)x = 0, \; l \neq 0$

1.4 Equations reducible to quadratic form

We now discuss different types of equations, which can be reduced to a quadratic equation by some proper substitution.

Type (i) The equations of the type $ax^4 + bx^2 + c = 0$

Replacing $x^2 = y$ in equation $ax^4 + bx^2 + c = 0$,

we get a quadratic equation in y.

Example 1: Solve the equation $x^4 - 13x^2 + 36 = 0$.

Solution: $x^4 - 13x^2 + 36 = 0$ (i)

Let $x^2 = y$. Then $x^4 = y^2$

Equation (i) becomes

$y^2 - 13y + 36 = 0$ which can be factorized as

$y^2 - 9y - 4y + 36 = 0$

$y(y - 9) - 4(y - 9) = 0$

$(y - 9)(y - 4) = 0$

Either $y - 9 = 0$ or $y - 4 = 0$, that is,

$y = 9$ or $y = 4$

Put $y = x^2$

$x^2 = 9$ or $x^2 = 4$

$\Rightarrow$ $x = \pm 3$ or $x = \pm 2$

$\therefore$ The solution set is $\{\pm 2, \pm 3\}$

Type (ii) The equations of the type $ap(x) + \dfrac{b}{p(x)} = c$

Example 2: Solve the equation $2(2x - 1) + \dfrac{3}{2x - 1} = 5$.

Solution: Given that $2(2x - 1) + \dfrac{3}{2x - 1} = 5$ (i)

Let $2x - 1 = y$.

Then the equation (i) becomes

$$2y + \frac{3}{y} = 5 \text{ or } 2y^2 + 3 = 5y$$

$\Rightarrow$ $2y^2 - 5y + 3 = 0$

Using quadratic formula

$$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$= \frac{5 \pm \sqrt{25-24}}{4} = \frac{5 \pm \sqrt{1}}{4} = \frac{5 \pm 1}{4}$$

We have $y = \frac{5+1}{4} = \frac{6}{4} = \frac{3}{2}$ or $y = \frac{5-1}{4} = \frac{4}{4} = 1$

When $y = \frac{3}{2}$,

$2x - 1 = \frac{3}{2}$ $\quad (\because\ y = 2x - 1)$

$\Rightarrow \quad 2x = \frac{3}{2} + 1 = \frac{5}{2} \quad \Rightarrow x = \frac{5}{4}$

When $y = 1$,

$2x - 1 = 1$ $\quad (\because\ y = 2x - 1)$

$\Rightarrow \quad 2x = 1 + 1 = 2 \quad \Rightarrow x = 1.$

Thus, the solution set is $\left\{1, \frac{5}{4}\right\}$.

Type (iii) Reciprocal equations of the type:

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0 \text{ or } ax^4 + bx^3 + cx^2 + bx + a = 0$$

An equation is said to be a **reciprocal equation**, if it remains unchanged, when x is replaced by $\frac{1}{x}$.

Replacing x by $\frac{1}{x}$ in $ax^4 - bx^3 + cx^2 - bx + a = 0$, we have

$a\left(\frac{1}{x}\right)^4 - b\left(\frac{1}{x}\right)^3 + c\left(\frac{1}{x}\right)^2 - b\left(\frac{1}{x}\right) + a = 0$ which is simplified as

$a - bx + cx^2 - bx^3 + ax^4 = 0$. We get the same equation.

Thus $ax^4 - bx^3 + cx^2 - bx + a = 0$ is a reciprocal equation.

The method for solving reciprocal equation is illustrated through an example.

Example 3: Solve the equation $2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$.

Solution: $2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$

Dividing each term by x^2

$$\frac{2x^4}{x^2} - \frac{5x^3}{x^2} - \frac{14x^2}{x^2} - \frac{5x}{x^2} + \frac{2}{x^2} = 0$$

$$2x^2 - 5x - 14 - \frac{5}{x} + \frac{2}{x^2} = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) - 14 = 0 \qquad \text{(i)}$$

Let $x + \frac{1}{x} = y$. Then $x^2 + \frac{1}{x^2} = y^2 - 2$

So equation (i) becomes $2(y^2 - 2) - 5y - 14 = 0$ or $2y^2 - 4 - 5y - 14 = 0$

$2y^2 - 5y - 18 = 0$

$2y^2 - 9y + 4y - 18 = 0$ or $y(2y - 9) + 2(2y - 9) = 0$

$\Rightarrow (2y - 9)(y + 2) = 0$

Either $2y - 9 = 0$ or $y + 2 = 0$

As $y = x + \frac{1}{x}$, so we have

$2\left(x + \frac{1}{x}\right) - 9 = 0$ or $x + \frac{1}{x} + 2 = 0$

$2x^2 - 9x + 2 = 0$ or $x^2 + 2x + 1 = 0$

By quadratic formula, we get

$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 2 \times 2}}{2 \times 2}$ or $x = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 1}}{2 \times 1}$

$\Rightarrow \quad x = \frac{9 \pm \sqrt{81 - 16}}{4}$ or $x = \frac{-2 \pm \sqrt{4 - 4}}{2}$

$= \frac{9 \pm \sqrt{65}}{4}$ or $x = \frac{-2 \pm 0}{2} \Rightarrow x = -1, -1$

Thus, the solution set is $\left\{-1, \frac{9 - \sqrt{65}}{4}, \frac{9 + \sqrt{65}}{4}\right\}$.

Type (iv) Exponential equations:

In **exponential equations,** variable occurs in exponent.

The method of solving such equations is illustrated through an example.

Example 4: Solve the equation $5^{1+x} + 5^{1-x} = 26$.

Solution: $5^{1+x} + 5^{1-x} = 26$

$5^1 . 5^x + 5^1 . 5^{-x} = 26$ or $5 . 5^x + \frac{5}{5^x} - 26 = 0$ (i)

Let $5^x = y$. Then equation (i) becomes

$5y + \frac{5}{y} - 26 = 0$

$5y^2 + 5 - 26y = 0$ or $5y^2 - 26y + 5 = 0$

$5y^2 - 25y - y + 5 = 0$

$5y(y - 5) - 1(y - 5) = 0$

$(y - 5)(5y - 1) = 0$

Either $y - 5 = 0$ or $5y - 1 = 0$, that is,

$$y = 5 \quad \text{or} \quad 5y = 1 \Rightarrow y = \frac{1}{5}$$

Put $\quad y = 5^x$

$$5^x = 5^1 \quad \text{or} \quad 5^x = 5^{-1} \Rightarrow x = 1 \quad \text{or} \quad x = -1$$

$\therefore$ The solution set is $\{\pm 1\}$.

Type (v) The equations of the type:

$$(x + a)(x + b)(x + c)(x + d) = k, \text{ where } a + b = c + d$$

Example 5: Solve the equation $(x - 1)(x + 2)(x + 8)(x + 5) = 19$.

Solution: $(x - 1)(x + 2)(x + 8)(x + 5) = 19$

or $\quad [(x - 1)(x + 8)][(x + 2)(x + 5)] - 19 = 0 \qquad (\because -1 + 8 = 2 + 5)$

$(x^2 + 7x - 8)(x^2 + 7x + 10) - 19 = 0 \qquad$ (i)

Let $\quad x^2 + 7x = y$

Then eq. (i) becomes

$$(y - 8)(y + 10) - 19 = 0$$
$$y^2 + 2y - 80 - 19 = 0$$
$$y^2 + 2y - 99 = 0$$
$$y^2 + 11y - 9y - 99 = 0$$
$$y(y + 11) - 9(y + 11) = 0$$
$$(y + 11)(y - 9) = 0$$

Either $\quad y + 11 = 0 \quad$ or $\quad y - 9 = 0$

Put $\quad y = x^2 + 7x$, so

$x^2 + 7x + 11 = 0 \quad$ or $\quad x^2 + 7x - 9 = 0$

Solving by quadratic formula, we have

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(11)}}{2(1)} \quad \text{or} \quad x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-9)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 - 44}}{2} = = \frac{-7 \pm \sqrt{5}}{2} \qquad = \frac{-7 \pm \sqrt{49 + 36}}{2} = \frac{-7 \pm \sqrt{85}}{2}$$

$\therefore$ The solution set is $\left\{\frac{-7 \pm \sqrt{5}}{2}, \frac{-7 \pm \sqrt{85}}{2}\right\}$.

EXERCISE 1.3

Solve the following equations.

1. $2x^4 - 11x^2 + 5 = 0$
2. $2x^4 = 9x^2 - 4$
3. $5x^{1/2} = 7x^{1/4} - 2$
4. $x^{2/3} + 54 = 15x^{1/3}$
5. $3x^{-2} + 5 = 8x^{-1}$
6. $(2x^2 + 1) + \frac{3}{2x^2 + 1} = 4$
7. $\frac{x}{x - 3} + 4\left(\frac{x - 3}{x}\right) = 4$
8. $\frac{4x + 1}{4x - 1} + \frac{4x - 1}{4x + 1} = 2\frac{1}{6}$
9. $\frac{x - a}{x + a} - \frac{x + a}{x - a} = \frac{7}{12}$
10. $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

11. $2x^4 + x^3 - 6x^2 + x + 2 = 0$ 12. $4.2^{2x+1} - 9.2^x + 1 = 0$

13. $3^{2x+2} = 12.3^x - 3$ 14. $2^x + 64.2^{-x} - 20 = 0$

15. $(x+1)(x+3)(x-5)(x-7) = 192$

16. $(x-1)(x-2)(x-8)(x+5) + 360 = 0$

1.5 Radical equations

An equation involving expression under the radical sign is called a **radical equation**.

e.g., $\sqrt{x+3} = x+1$ and $\sqrt{x-1} = \sqrt{x-2} + 1$

1.5 (i) Equations of the type: $\sqrt{ax+b} = cx+d$

Example 1: Solve the equation $\sqrt{3x+7} = 2x+3$.

Solution: $\sqrt{3x+7} = 2x+3$ (i)

Squaring both sides of the equation (i), we get

$$(\sqrt{3x+7})^2 = (2x+3)^2$$

or $3x + 7 = 4x^2 + 12x + 9$

Simplifying the above equation, we have

$$4x^2 + 9x + 2 = 0$$

Applying quadratic formula,

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4\times4\times2}}{2\times4}$$

$$= \frac{-9 \pm \sqrt{81-32}}{8} = \frac{-9 \pm \sqrt{49}}{8} = \frac{-9 \pm 7}{8}$$

$\therefore$ $x = \frac{-9+7}{8} = \frac{-2}{8} = \frac{-1}{4}$

or $x = \frac{-9-7}{8} = \frac{-16}{8} = -2$

> **Note:** Extraneous root is introduced either by squaring the given equation or clearing it of fractions.

Checking:

Putting $x = -\frac{1}{4}$ in the equation (i), we have

$$\sqrt{3\left(-\frac{1}{4}\right) + 7} = 2\left(-\frac{1}{4}\right) + 3 \Rightarrow \sqrt{\frac{-3+28}{4}} = -\frac{1}{2} + 3 \Rightarrow \sqrt{\frac{25}{4}} = \frac{5}{2}$$ which is true.

Putting $x = -2$ in the equation (i), we get

$$\sqrt{3(-2)+7} = 2(-2)+3 \quad \Rightarrow \sqrt{1} = -1 \text{ which is not true.}$$

On checking, we find that $x = -2$ does not satisfy the equation (i), so it is an extraneous root. Thus the solution set is $\left\{-\frac{1}{4}\right\}$.

1.5 (ii) Equations of the type $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Example 2: Solve the equation $\sqrt{x+3} + \sqrt{x+6} = \sqrt{x+11}$.

Solution:

$$\sqrt{x+3} + \sqrt{x+6} = \sqrt{x+11} \qquad \text{(i)}$$

Squaring both sides of the equation (i), we have

$$x+3+x+6+2\left(\sqrt{x+3}\right)\left(\sqrt{x+6}\right) = x+11$$

or $$2\sqrt{x^2+9x+18} = -x+2 \qquad \text{(ii)}$$

Squaring both sides of the equation (ii), we get

$$4\,(x^2+9x+18) = x^2-4x+4$$

or $$3x^2+40x+68 = 0$$

Applying quadratic formula, we get

$$x = \frac{-40 \pm \sqrt{(40)^2 - 4\times 3\times 68}}{2\times 3} = \frac{-40 \pm \sqrt{1600-816}}{6}$$

$$= \frac{-40 \pm \sqrt{784}}{6} = \frac{-40 \pm 28}{6}$$

We have $x = \frac{-40+28}{6} = \frac{-12}{6} = -2$ or $x = \frac{-40-28}{6} = \frac{-68}{6} = \frac{-34}{3}$

Checking: Putting $x = \frac{-34}{3}$ in the equation (i), we have

$$\sqrt{\frac{-34}{3}+3} + \sqrt{\frac{-34}{3}+6} = \sqrt{\frac{-34}{3}+11}$$

or $$\sqrt{\frac{-34+9}{3}} + \sqrt{\frac{-34+18}{3}} = \sqrt{\frac{-34+33}{3}}$$

$\Rightarrow$ $$\sqrt{\frac{25}{3}\times(-1)} + \sqrt{\frac{16}{3}\times(-1)} = \sqrt{\frac{1}{3}\times(-1)}$$

$$\Rightarrow \frac{5}{\sqrt{3}}i + \frac{4}{\sqrt{3}}i = \frac{1}{\sqrt{3}}i \text{ which is not true.}$$

As $\frac{-34}{3}$ is extraneous root, so the solution set is $\{-2\}$.

1.5(iii) Equations of the type: $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 3: Solve the equation $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$.

Solution: $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$

Let $x^2-3x=y$

Then $\sqrt{y+36}-\sqrt{y+9}=3$

Squaring both sides, we get

$$y+36+y+9-2\left(\sqrt{y+36}\right)\left(\sqrt{y+9}\right)=9$$

$$2y+45-2\sqrt{(y+36)\,(y+9)}=9$$

$$-2\sqrt{y^2+45y+324}=-2y-36 \quad \text{or} \quad -2\sqrt{y^2+45y+324}=-2(y+18)$$

$\Rightarrow \quad \sqrt{y^2+45y+324}=y+18$

Again squaring both sides, we get

$$y^2+45y+324=y^2+36y+324$$

$9y=0 \quad \Rightarrow \quad y=0$

As $x^2-3x=y$, so $x^2-3x=0$

$\Rightarrow \quad x(x-3)=0$

Either $x=0$ or $x-3=0 \Rightarrow x=3$

$\therefore$ $x=0$, 3 are the roots of the equation.

Thus, the solution set is $\{0, 3\}$.

EXERCISE 1.4

Solve the following equations.

1. $2x+5=\sqrt{7x+16}$
2. $\sqrt{x+3}=3x-1$
3. $4x=\sqrt{13x+14}-3$
4. $\sqrt{3x+100}-x=4$
5. $\sqrt{x+5}+\sqrt{x+21}=\sqrt{x+60}$
6. $\sqrt{x+1}+\sqrt{x-2}=\sqrt{x+6}$
7. $\sqrt{11-x}-\sqrt{6-x}=\sqrt{27-x}$
8. $\sqrt{4a+x}-\sqrt{a-x}=\sqrt{a}$
9. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$
10. $\sqrt{x^2+3x+8}+\sqrt{x^2+3x+2}=3$
11. $\sqrt{x^2+3x+9}+\sqrt{x^2+3x+4}=5$

MISCELLANEOUS EXERCISE - 1

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) Standard form of quadratic equation is

(a) $bx + c = 0\ , b \neq 0$ (b) $ax^2 + bx + c = 0\ , a \neq 0$

(c) $ax^2 = bx\ , a \neq 0$ (d) $ax^2 = 0\ , a \neq 0$

(ii) The number of terms in a standard quadratic equation $ax^2 + bx + c = 0$ is

(a) 1 (b) 2 (c) 3 (d) 4

(iii) The number of methods to solve a quadratic equation is

(a) 1 (b) 2 (c) 3 (d) 4

(iv) The quadratic formula is

(a) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (b) $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $x = \dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$ (d) $x = \dfrac{b \pm \sqrt{b^2 + 4ac}}{2a}$

(v) Two linear factors of $x^2 - 15x + 56$ are

(a) $(x - 7)$ and $(x + 8)$ (b) $(x + 7)$ and $(x - 8)$

(c) $(x - 7)$ and $(x - 8)$ (d) $(x + 7)$ and $(x + 8)$

(vi) An equation, which remains unchanged when x is replaced by $\dfrac{1}{x}$ is called a/an

(a) Exponential equation (b) Reciprocal equation

(c) Radical equation (d) None of these

(vii) An equation of the type $3^x + 3^{2-x} + 6 = 0$ is a/an

(a) Exponential equation (b) Radical equation

(c) Reciprocal equation (d) None of these

(viii) The solution set of equation $4x^2 - 16 = 0$ is

(a) $\{\pm 4\}$ (b) $\{4\}$ (c) $\{\pm 2\}$ (d) ± 2

(ix) An equation of the form $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an

(a) Reciprocal equation (b) Radical equation

(c) Exponential equation (d) None of these

2. Write short answers of the following questions.

(i) Solve $x^2 + 2x - 2 = 0$

(ii) Solve by factorization $5x^2 = 15x$

(iii) Write in standard form $\dfrac{1}{x+4} + \dfrac{1}{x-4} = 3$

(iv) Write the names of the methods for solving a quadratic equation.

(v) Solve $\left(2x - \dfrac{1}{2}\right)^2 = \dfrac{9}{4}$ (vi) Solve $\sqrt{3x + 18} = x$

(vii) Define quadratic equation.

(viii) Define reciprocal equation.

(ix) Define exponential equation.

(x) Define radical equation.

3. Fill in the blanks

(i) The standard form of the quadratic equation is________.

(ii) The number of methods to solve a quadratic equation are ________.

(iii) The name of the method to derive a quadratic formula is ________.

(iv) The solution of the equation $ax^2 + bx + c = 0, a \neq 0$ is ________.

(v) The solution set of $25x^2 - 1 = 0$ is ________.

(vi) An equation of the form $2^{2x} - 3.2^x + 5 = 0$ is called a/an ________ equation.

(vii) The solution set of the equation $x^2 - 9 = 0$ is ________.

(viii) An equation of the type $x^4 + x^3 + x^2 + x + 1 = 0$ called a/an ________ equation.

(ix) A root of an equation, which do not satisfy the equation is called ________ root.

(x) An equation involving impression of the variable under ________ is called radical equation.

SUMMARY

- An equation which contains the square of the unknown (variable) quantity, but no higher power, is called a **quadratic equation** or an equation of the **second degree.**
- A **second degree equation** in one variable x, $ax^2 + bx + c = 0$ where $a \neq 0$ and a, b, c are real numbers, is called the **general or standard form** of a quadratic equation.
- An equation is said to be a **reciprocal equation,** if it remains unchanged, when x is replaced by $\frac{1}{x}$.
- In **exponential equations,** variables occur in exponents.
- An equation involving expression under the **radical sign** is called a **radical equation.**
- Quadratic formula for $ax^2 + bx + c = 0, a \neq 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Any quadratic equation is solved by the following three methods.

(i) Factorization (ii) Completing square

(iii) Quadratic formula