Unit-10

TANGENT TO A CIRCLE

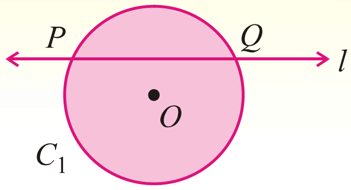
In this unit, students will learn how to

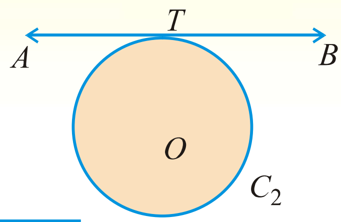
Prove the following theorems alongwith corollaries and apply them to solve appropriate problems.

- *If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.*
- *The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.*
- *The two tangents drawn to a circle from a point outside it, are equal in length.*
- *If two circles touch externally or internally, the distance between their centres is respectively equal to the sum or difference of their radii.*

Definition: A secant is a straight line which cuts the circumference of a circle in two distinct points. In the figure l indicates the secant line to the circle C_1.

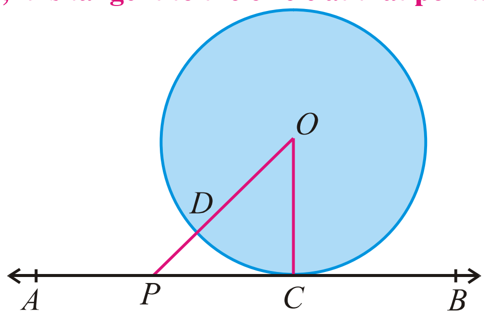
Definition: A tangent to a circle is the straight line which touches the circumference at a single point only. The point of tangency is also known as the point of contact. In the figure $\overleftrightarrow{AB}$ indicates the tangent line to the circle C_2.





THEOREM 1

10.1(*i*) If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.



Given: A circle with centre O and $\overline{OC}$ is the radial segment. $\overleftrightarrow{AB}$ is perpendicular to $\overline{OC}$ at its outer end C.

To prove: $\overleftrightarrow{AB}$ is a tangent to the circle at C.

Construction: Take any point P other than C on $\overleftrightarrow{AB}$. join O with P.

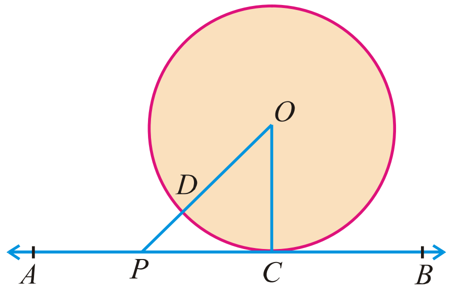
Proof:

Statements	Reasons
In ΔOCP,	
$m\angle OCP = 90^o$	$\overleftrightarrow{AB} \perp \overline{OC}$ (given)
and $m\angle OPC < 90^o$	Acute angle of right angled triangle.
$m\overline{OP} > m\overline{OC}$	Greater angle has greater side opposite to it.

$\therefore$ P is a point outside the circle. Similarly, every point on $\overleftrightarrow{AB}$ except C lies outside the circle. Hence $\overleftrightarrow{AB}$ intersects the circle at one point C only. *i.e.*, $\overleftrightarrow{AB}$ is a tangent to the circle at one point only.	$\overline{OC}$ is the radial segment.

THEOREM 2

10.1(*ii*) The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.



Given: In a circle with centre O has radius $\overline{OC}$, $\overleftrightarrow{AB}$ is the tangent to the circle at point C.

To prove: $\overleftrightarrow{AB}$ and radial segment $\overline{OC}$ are perpendicular to each other.

Construction: Take any point P other than C on the tangent line $\overleftrightarrow{AB}$. Join O with P so that $\overline{OP}$ meets the circle at D.

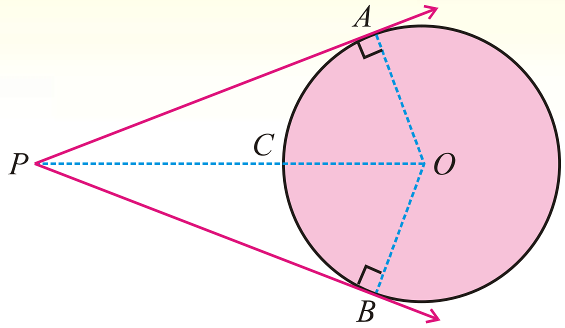
Proof:

Statements	Reasons
$\overleftrightarrow{AB}$ is the tangent to the circle at point C. Whereas	Given
$\overline{OP}$ cuts the circle at D.	Construction
$\therefore$ $m\overline{OC} = m\overline{OD}$ (i)	Radii of the same circle
But $m\overline{OD} < m\,\overline{OP}$ (ii)	Point P is outside the circle.
$\therefore$ $m\overline{OC} < m\overline{OP}$	Using (i) and (ii)
So radius $\overline{OC}$ is shortest of all lines that can be drawn from O to the tangent line $\overleftrightarrow{AB}$	
Also $\overline{OC} \perp \overleftrightarrow{AB}$	
Hence, radial segment $\overline{OC}$ is perpendicular to the tangent $\overleftrightarrow{AB}$.	

Corollary: There can only be one perpendicular drawn to the radial segment $\overline{OC}$ at the point C of the circle. It follows that one and only one tangent can be drawn to the circle at the given point C on its circumference.

THEOREM 3

10.1(*iii*) Two tangents drawn to a circle from a point outside it, are equal in length.



Given: Two tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn from an external point P to the circle with centre O.

To prove: $m\overline{PA} = m\overline{PB}$

Construction: Join O with A , B and P, so that we form $\angle rt\Delta^s$ OAP and OBP.

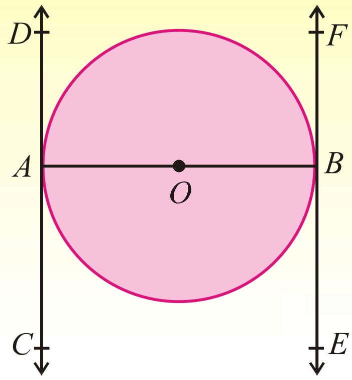
Proof:

Statements	Reasons
In $\angle rt\ \Delta^s\ OAP \leftrightarrow OBP$	
$m\angle OAP = m\angle OBP = 90^\circ$	Radii $\perp$ to the tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$
hyp. $\overline{OP}$ = hyp. $\overline{OP}$	Common
$m\ \overline{OA} = m\ \overline{OB}$	Radii of the same circle.
$\therefore\ \Delta\ OAP \cong \Delta\ OBP$	In $\angle rt\ \Delta^s$ H.S $\cong$ H.S
Hence, $m\overline{PA} = m\overline{PB}$	

Note: The length of a tangent to a circle is measured from the given point to the point of contact.

Corollary: If O is the centre of a circle and two tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn from an external point P then $\overline{OP}$ is the right bisector of the chord of contact $\overline{AB}$.

Example 1: $\overline{AB}$ is a diameter of a given circle with centre O. Tangents are drawn at the end points A and B. Show that the two tangents are parallel.



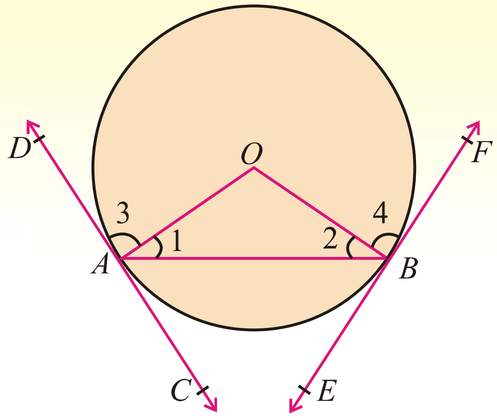
Given: $\overline{AB}$ is a diameter of a given circle with centre O.
CD is the tangent to the circle at point A
and EF is an other tangent at point B.

To prove: $\overline{CD} \parallel \overline{EF}$

Proof:

Statements	Reasons
$\overline{AB}$ is the diameter of a circle with centre O. $\therefore$ $\overline{OA}$ and $\overline{OB}$ are radii of the same circle.	Given
Moreover $\overleftrightarrow{CD}$ is a tangent to the circle at A.	Given
$\therefore$ $\overline{OA} \perp \overleftrightarrow{CD}$	By Theorem 1
$\overline{AB} \perp \overleftrightarrow{CD}$ (i)	
Similarly $\overleftrightarrow{EF}$ is tangent at point B.	Given
So $\overline{OB} \perp \overleftrightarrow{EF}$	By Theorem 1
$\Rightarrow$ $\overrightarrow{AB} \perp \overrightarrow{EF}$ (ii)	
Hence $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$	From (i) and (ii) ($\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ are perpendicular to $\overline{AB}$)

Example 2: In a circle, the tangents drawn at the ends of a chord, make equal angles with that chord.



Given: $\overline{AB}$ is the chord of a circle with centre O.

$\overleftrightarrow{CAD}$ is the tangent at point A and $\overleftrightarrow{EBF}$ is an other tangent at point B.

To prove: $m\angle BAD = m\angle ABF$

Construction: Join O with A and B so that we form a ΔOAB then write $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ as shown in the figure.

Proof:

Statements		Reasons
In $\Delta\ OAB$		Construction
$\because\ m\overline{OA} = m\overline{OB}$		Radii of the same circle.
$\therefore\ m\angle 1 = m\angle 2$	(i)	Angles opp. to equal sides of ΔOAB
Also $\because\ \overline{OA} \perp \overleftrightarrow{CD}$		Radius is $\perp$ to the tangent line
$\therefore\ m\angle 3 = m\angle OAD = 90^{o}$	(ii)	
Similarly $\overline{OB} \perp \overleftrightarrow{EF}$		Radius is $\perp$ to the tangent
$\therefore\ m\angle 4 = m\angle OBF = 90^{o}$	(iii)	
Hence $m\angle 3 = m\angle 4$	(iv)	Using (ii) and (iii)
$\Rightarrow\ m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$		Adding (i) and (iv)
i.e., $m\angle BAD = m\angle ABF$		

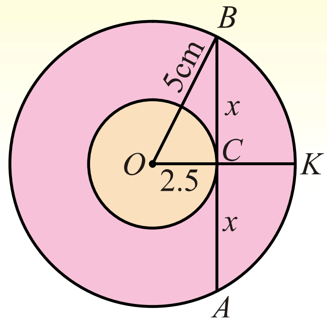
EXERCISE 10.1

1. Prove that the tangents drawn at the ends of a diameter in a given circle must be parallel.

2. The diameters of two concentric circles are 10cm and 5cm respectively. Look for the length of any chord of the outer circle which touches the inner one.

   (Hint) From the figure

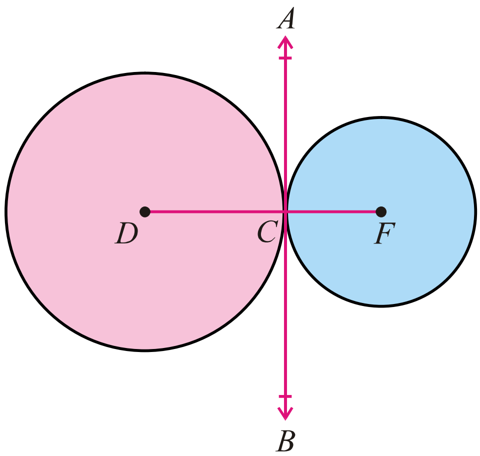
   $m\overline{AB} = 2x = 2\sqrt{25 - 6.25} = 2\sqrt{18.75} \simeq 8.7\text{cm}$



3. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the common tangents drawn to the pair of circles. If A and C are the points of tangency of 1st circle where B and D are the points of tangency of 2nd circle, then prove that $\overline{AC} \parallel \overline{BD}$.

THEOREM 4(A)

10.1(iv) If two circles touch externally then the distance between their centres is equal to the sum of their radii.



Given: Two circles with centres D and F respectively touch each other externally at point C. So that $\overline{CD}$ and $\overline{CF}$ are respectively the radii of the two circles.

To prove: Point C lies on the join of centres D and F and $m\overline{DF} = m\overline{DC} + m\overline{CF}$

Construction: Draw $\overleftrightarrow{ACB}$ as a common tangent to the pair of circles at C.

Proof:

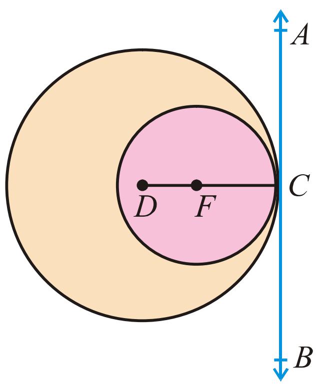
Statements	Reasons
Both circles touch externally at C whereas $\overline{CD}$ is radial segment and $\overleftrightarrow{ACB}$ is the common tangent.	
$\therefore \quad m\angle ACD = 90^o$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overline{AB}$
Similarly $\overline{CF}$ is radial segment and $\overleftrightarrow{ACB}$ is the common tangent	
$\therefore \quad m\angle ACF = 90^o$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$
$m\angle ACD + m\angle ACF = 90^o + 90^o$	Adding (i) and (ii)
$m\angle DCF = 180^o$ (iii)	Sum of supplementary adjacent angles
Hence DCF is a line segment with point C between D and F	
and $\quad m\overline{DF} = m\overline{DC} + m\overline{CF}$	

EXERCISE 10.2

1. $\overline{AB}$ and $\overline{CD}$ are two equal chords in a circle with centre O. H and K are respectively the mid points of the chords. Prove that $\overline{HK}$ makes equal angles with $\overline{AB}$ and $\overline{CD}$.
2. The radius of a circle is 2.5 cm. $\overline{AB}$ and $\overline{CD}$ are two chords 3.9cm apart. If $m\overline{AB}$ = 1.4 cm, then measure the other chord.
3. The radii of two intersecting circles are 10cm and 8cm. If the length of their common chord is 6cm then find the distance between the centres.
4. Show that greatest chord in a circle is its diameter.

THEOREM 4(B)

10.1(v) If two circles touch each other internally, then the point of contact lies on the line segment through their centres and distance between their centres is equal to the difference of their radii.



Given: Two circles with centres D and F touch each other internally at point C.

So that $\overline{CD}$ and $\overline{CF}$ are the radii of two circles.

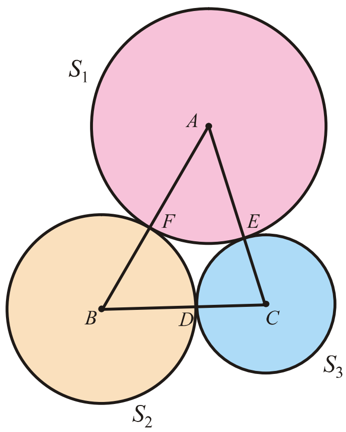
To prove: Point C lies on the join of centres D and F extended and $m\overline{DF} = m\overline{DC} - m\overline{CF}$

Construction: Draw $\overleftrightarrow{ACB}$ as the common tangent to the pair of circles at C.

Proof:

Statements	Reasons
Both circles touch internally at C whereas $\overleftrightarrow{ACB}$ is the common tangent and $\overline{CD}$ is the radial segment of the first circle.	
$\therefore \quad m\angle ACD = 90^o \quad$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overline{AB}$
Similarly $\overleftrightarrow{ACB}$ is the common tangent and $\overline{CF}$ is the radial segment of the second circle.	
$\therefore \quad m\angle ACF = 90^o \quad$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$.
$\Rightarrow \quad m\angle ACD = m\angle ACF = 90^o$	Using (i) and (ii)
Where $\angle ACD$ and $\angle ACF$ coincide each other with point F between D and C.	
Hence $\quad m\overline{DC} = m\overline{DF} + m\overline{FC}$	
i.e., $\quad m\overline{DC} - m\overline{FC} = m\overline{DF}$	
or $\quad m\overline{DF} = m\overline{DC} - m\overline{FC}$	

Example 1: Three circles touch in pairs externally. Prove that the perimeter of a triangle formed by joining centres is equal to the sum of their diameters.



Given: Three circles have centres A, B and C their radii are r_1, r_2 and r_3 respectively. They touch in pairs externally at D, E and F. So that ΔABC is formed by joining the centres of these circles.

To prove:

Perimeter of $\Delta ABC = 2r_1 + 2r_2 + 2r_3 = d_1 + d_2 + d_3$

= Sum of the diameters of these circles.

Proof:

Statements	Reasons
Three circles with centres A, B and C touch in pairs externally at the points, D, E and F.	Given
$\therefore \quad m\overline{AB} = m\overline{AF} + m\overline{FB}$ (i)	
$m\overline{BC} = m\overline{BD} + m\overline{DC}$ (ii)	
and $\quad m\overline{CA} = m\overline{CE} + m\overline{EA}$ (iii)	
$m\overline{AB} + m\overline{BC} + m\overline{CA} = m\overline{AF} + m\overline{FB} + m\overline{BD} + m\overline{DC} + m\overline{CE} + m\overline{EA}$ $= (m\overline{AF} + m\overline{EA}) + (m\overline{FB} + m\overline{BD}) + (m\overline{CD} + m\overline{CE})$	Adding (i), (ii) and (iii)
Perimeter of $\Delta ABC = 2r_1 + 2r_2 + 2r_3$ $= d_1 + d_2 + d_3$ = Sum of diameters of the circles.	$d_1 = 2r_1$, $d_2 = 2r_2$ and $d_3 = 2r_3$ are diameters of the circles.

EXERCISE 10.3

1. Two circles with radii 5cm and 4cm touch each other externally. Draw another circle with radius 2.5cm touching the first pair, externally.
2. If the distance between the centres of two circles is the sum or the difference of their radii they will touch each other.

MISCELLANEOUS EXERCISE 10

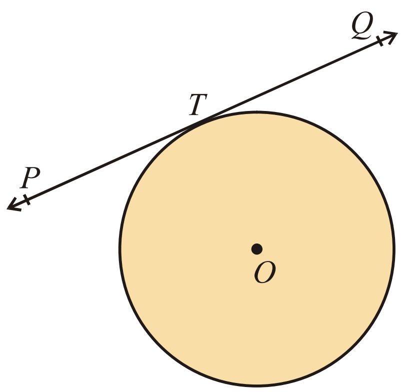
Multiple Choice Questions

1. Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) In the adjacent figure of the circle, the line $\overleftrightarrow{PTQ}$ is named as

(a) an arc (b) a chord

(c) a tangent (d) a secant

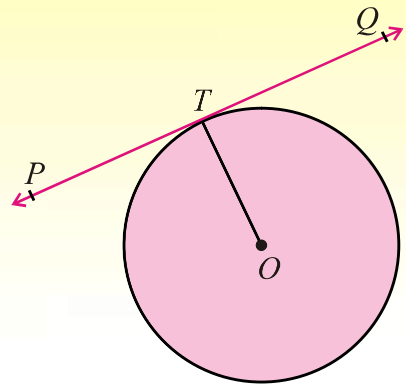


(ii) In a circle with centre O, if $\overline{OT}$ is the radial segment and $\overleftrightarrow{PTQ}$ is the tangent line, then

(a) $\overline{OT} \perp \overleftrightarrow{PQ}$ (b) $\overline{OT} \not\perp \overleftrightarrow{PQ}$

(c) $\overline{OT} \parallel \overleftrightarrow{PQ}$

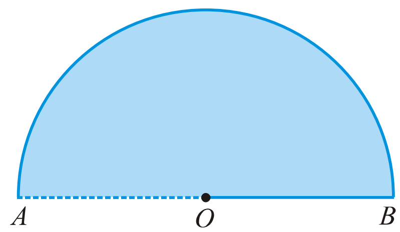
(d) $\overline{OT}$ is right bisector of $\overleftrightarrow{PQ}$



(iii) In the adjacent figure, find semicircular area if $\pi \simeq 3.1416$ and $m\overline{OA} = 20$cm.

(a) 62.83sq cm (b) 314.16sq cm

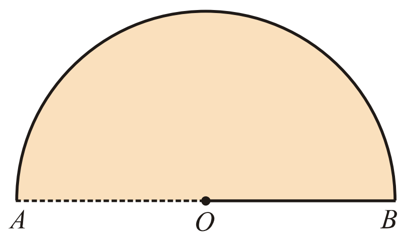
(c) 436.20sq cm (d) 628.32sq cm



(iv) In the adjacent figure, find half the perimeter of circle with centre O if $\pi \simeq 3.1416$ and $m\overline{OA} = 20$cm.

(a) 31.42 cm (b) 62.832 cm

(c) 125.65 cm (d) 188.50 cm



(v) A line which has two points in common with a circle is called:

(a) sine of a circle (b) cosine of a circle

(c) tangent of a circle (d) secant of a circle

(vi) A line which has only one point in common with a circle is called:

(a) sine of a circle (b) cosine of a circle

(c) tangent of a circle (d) secant of a circle

(vii) Two tangents drawn to a circle from a point outside it are of .... in length.

(a) half (b) equal (c) double (d) triple

(viii) A circle has only one:

(a) secant (b) chord (c) diameter (d) centre

(ix) A tangent line intersects the circle at:

(a) three points (b) two points (c) single point (d) no point at all

(x) Tangents drawn at the ends of diameter of a circle are .... to each other.

(*a*) parallel (*b*) non-parallel (*c*) collinear (*d*) perpendicular

(xi) The distance between the centres of two congruent touching circles externally is:

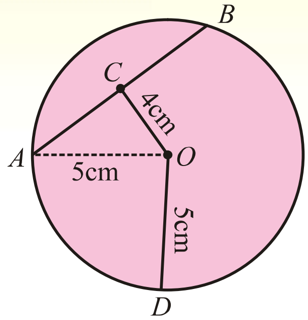
(*a*) of zero length (*b*) the radius of each circle

(*c*) the diameter of each circle (*d*) twice the diameter of each circle

(xii) In the adjacent circular figure with centre O and radius 5cm, the length of the chord intercepted at 4cm away from the centre of this circle is:

(*a*) 4cm (*b*) 6cm

(*c*) 7cm (*d*) 9cm

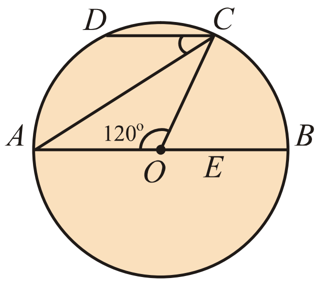


(xiii) In the adjoining figure, there is a circle with centre O.

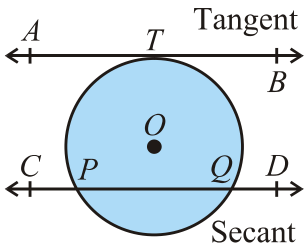
If $\overline{DC}$ // diameter $\overline{AB}$ and $m\angle AOC = 120^o$, then $m\angle ACD$ is:

(*a*) 40^o (*b*) 30^o

(*c*) 50^o (*d*) 60^o



SUMMARY



- A **secant** is a straight line which cuts the circumference of a circle in two distinct points. In the figure, the secant $\overleftrightarrow{CD}$ cuts the circle at two distinct points P and Q.
- A **tangent** to a circle is the straight line which touches the circumference at one point only. The point of tangency is also known as the point of contact in the figure. $\overleftrightarrow{AB}$ is the tangent line to the circle at the point T.

- The **length of a tangent** to a circle is measured from the given point to the point of contact.
- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside it, are equal in length.
- If two circles touch externally or internally, the distance between their centres is respectively equal to the sum or difference of their radii.