Unit-3

VARIATIONS

In this unit, students will learn how to

- *define ratio, proportions and variations (direct and inverse).*
- *find 3rd, 4th, mean and continued proportion.*
- *apply theorems of invertendo, alternendo, componendo, dividendo and componendo & dividendo to find proportions.*
- *define joint variation.*
- *solve problems related to joint variation.*
- *use k-method to prove conditional equalities involving proportions.*
- *solve real life problems based on variations.*

3.1 Ratio, Proportions and Variations

3.1(i) Define (*a*) ratio, (*b*) proportion and (*c*) variations (direct and inverse).

(a) Ratio

A relation between two quantities of the same kind (measured in same unit) is called **ratio**. If a and b are two quantities of the same kind and b is not zero, then the ratio of a and b is written as $a : b$ or in fraction $\frac{a}{b}$

e.g., if a hockey team wins 4 games and loses 5, then the ratio of the games won to games lost is 4 : 5 or in fraction $\frac{4}{5}$

Remember that:

(*i*) The **order** of the elements in a ratio is important.

(*ii*) In ratio $a : b$, the first term a is called **antecedent** and the second term b is called **consequent**.

(*iii*) A ratio has no **units**.

Example 1: Find the ratio of

(*i*) 200gm to 700 gm (*ii*) 1km to 600m

Solution: (*i*) Ratio of 200gm to 700 gm

$$200 : 700 = \frac{200}{700} = \frac{2}{7} = 2 : 7$$

Where 2 : 7 is the simplest (lowest) form of the ratio 200 : 700.

(*ii*) Ratio of 1km to 600m

Since 1km = 1000m

then $1000 : 600 = \frac{1000}{600} = \frac{10}{6} = \frac{5}{3} = 5 : 3$

or 1km : 600m = 1000:600

$$= \frac{1000}{100} : \frac{600}{100} = 10{:}6 = 5{:}3$$

Example 2: Find a, if the ratios $a + 3 : 7 + a$ and $4 : 5$ are equal.

Solution: Since the ratios $a + 3 : 7 + a$ and $4 : 5$ are equal.

$\therefore$ in fraction form

$$\frac{a+3}{7+a} = \frac{4}{5}$$

$$5(a + 3) = 4(7 + a)$$

$$5a + 15 = 28 + 4a$$

$$5a - 4a = 28 - 15$$

$$a = 13$$

Thus the given ratios will be equal if $a = 13$.

Example 3: If 2 is added in each number of the ratio 3 : 4, we get a new ratio 5 : 6. Find the numbers.

Solution: Because the ratio of two numbers is 3 : 4.

Multiply each number of the ratio with x. Then the numbers be $3x$, $4x$ and the ratio becomes $3x{:}4x$. Now according to the given condition

$$\frac{3x+2}{4x+2} = \frac{5}{6}$$

$$6(3x+2) = 5(4x+2) \quad \Rightarrow \quad 18x + 12 = 20x + 10$$

$$18x - 20x = 10 - 12 \quad \Rightarrow \quad -2x = -2 \quad \Rightarrow \quad x = 1$$

Thus the required numbers are

$$3x = 3(1) = 3$$

and $\quad 4x = 4(1) = 4.$

Example 4: Find the ratio $3a + 4b : 5a + 7b$ if $a : b = 5 : 8$.

Solution: Given that $a : b = 5 : 8$ or $\frac{a}{b} = \frac{5}{8}$

Now $\quad 3a + 4b : 5a + 7b = \dfrac{3a+4b}{5a+7b}$

$$= \frac{\frac{3a+4b}{b}}{\frac{5a+7b}{b}} = \frac{3\left(\frac{a}{b}\right) + 4\left(\frac{b}{b}\right)}{5\left(\frac{a}{b}\right) + 7\left(\frac{b}{b}\right)} \quad \text{(Dividing numerator and denominator by } b\text{)}$$

$$= \frac{3\left(\frac{5}{8}\right) + 4(1)}{5\left(\frac{5}{8}\right) + 7(1)} \qquad \left(\because \frac{a}{b} = \frac{5}{8}\right)$$

$$= \frac{\frac{15}{8} + 4}{\frac{25}{8} + 7} = \frac{\frac{15+32}{8}}{\frac{25+56}{8}} = \frac{47}{81}$$

Hence, $\quad 3a + 4b : 5a + 7b = 47 : 81.$

(b) Proportion

A **proportion** is a statement, which is expressed as an equivalence of two ratios.

If two ratios $a : b$ and $c : d$ are equal, then we can write $a : b = c : d$

Where quantities a, d are called **extremes**, while b, c are called **means**.

Symbolically the proportion of a, b, c and d is written as

$$a : b :: c : d$$

or $\quad a : b = c : d$

or $\frac{a}{b} = \frac{c}{d}$

i.e., $ad = bc$

This shows that, **Product of extremes = Product of means.**

Example 5: Find x, if 60m : 90m :: 20kg : x kg

Solution: Given that 60m : 90m :: 20kg : x kg

$60 : 90 = 20 : x$

∵ Product of extremes = Product of means

∴ $60x = 90 \times 20$

$x = \frac{90 \times 20}{60} = 30$ *i.e.,* x is 30 kg

Example 6: Find the cost of 15kg of sugar, if 7 kg of sugar costs 560 rupees.

Solution: Let the cost of 15kg of sugar be x-rupees.

Then in proportion form

15kg : 7kg :: Rs. x : Rs. 560

$15 : 7 = x : 560$

∵ Product of extremes = Product of means

∴ $15 \times 560 = 7x$

$7x = 15 \times 560$

$x = \frac{15 \times 560}{7} = 15(80) = 1200$

Thus, x = Rs. 1200.

EXERCISE 3.1

1. Express the following as a ratio $a : b$ and as a fraction in its simplest (lowest) form.
   (*i*) Rs. 750 , Rs. 1250 (*ii*) 450cm , 3 m
   (*iii*) 4kg , 2kg 750gm (*iv*) 27min. 30 sec, 1 hour
   (*v*) 75^o, 225^o
2. In a class of 60 students, 25 students are girls and remaining students are boys. Compute the ratio of
   (*i*) boys to total students (*ii*) boys to girls
3. If $3(4x - 5y) = 2x - 7y$, find the ratio $x : y$.
4. Find the value of p, if the ratios $2p + 5 : 3p + 4$ and $3 : 4$ are equal.
5. If the ratios $3x + 1 : 6 + 4x$ and $2 : 5$ are equal. Find the value of x.

6. Two numbers are in the ratio 5 : 8. If 9 is added to each number, we get a new ratio 8 : 11. Find the numbers.

7. If 10 is added in each number of the ratio 4 :13, we get a new ratio 1 : 2. What are the numbers?

8. Find the cost of 8kg of mangoes, if 5kg of mangoes cost Rs. 250.

9. If $a : b = 7 : 6$, find the value of $3a + 5b : 7b - 5a$.

10. Complete the following:

(*i*) If $\frac{24}{7} = \frac{6}{x}$, then $4x =$ __________

(*ii*) If $\frac{5a}{3x} = \frac{15b}{y}$, then $ay =$ __________

(*iii*) If $\frac{9pq}{2lm} = \frac{18p}{5m}$, then $5q =$ __________

11. Find x in the following proportions.

(*i*) $3x - 2 : 4 :: 2x + 3 : 7$ (*ii*) $\frac{3x-1}{7} : \frac{3}{5} :: \frac{2x}{3} : \frac{7}{5}$

(*iii*) $\frac{x-3}{2} : \frac{5}{x-1} :: \frac{x-1}{3} : \frac{4}{x+4}$ (*iv*) $p^2 + pq + q^2 : x :: \frac{p^3 - q^3}{p+q} : (p-q)^2$

(*v*) $8 - x : 11 - x :: 16 - x : 25 - x$

(c) Variation:

The word variation is frequently used in all sciences. There are two types of variations: (i) Direct variation (ii) Inverse variation.

(i) Direct Variation

If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) in the other quantity, then this variation is called **direct variation**.

In otherwords, if a quantity y varies directly with regard to a quantity x. We say that y is **directly proportional** to x and is written as $y \propto x$ or $y = kx$. *i.e.*, $\frac{y}{x} = k$, $k \neq 0$.

The sign $\propto$ read as "varies as" is called the sign of proportionality or variation, while $k \neq 0$ is known as constant of variation.

e.g., (*i*) Faster the speed of a car, longer the distance it covers.

(*ii*) The smaller the radius of the circle, smaller the circumference is.

Example 1: Find the relation between distance d of a body falling from rest varies directly as the square of the time t, neglecting air resistance. Find k, if $d = 16$ feet for $t = 1$ sec. Also derive a relation between d and t.

Solution: Since d is the distance of the body falling from rest in time t.

Then under the given condition

$$d \propto t^2$$

i.e., $d = kt^2$ (i)

Since $d = 16$ feet and $t = 1$ sec

Then equation (i) becomes

$16 = k(1)^2$

i.e., $k = 16$

put in eq. (i) $d = 16t^2$

Which is a relationship between the distance d and time t.

Activity:

From the above example:

(*i*) Find time t, when $d = 64$ feet

(*ii*) Find distance d, when $t = 3$ sec

Example 2: If y varies directly as x, find

(a) the equation connecting x and y.

(b) the constant of variation k and the relation between x and y, when $x = 7$ and $y = 6$

(c) the value of y, when $x = 21$.

Solution: (a) Given that y varies directly as x.

Therefore $y \propto x$, *i.e.,* $y = kx$, where k is constant of variation.

(b) Putting $x = 7$ and $y = 6$ in equation

$y = kx$ (i)

We get $6 = 7k \Rightarrow k = \frac{6}{7}$

Put in eq. (i) $y = \frac{6}{7}x$ (ii)

(c) Now put $x = 21$, in equation (ii)

Then $y = \frac{6}{7}(21) = 18$

Example 3: Given that A varies directly as the square of r and $A = \frac{1782}{7}$ cm^2, when $r = 9$cm. If $r = 14$cm, then find A.

Solution: Since A varies directly as square of r

$\therefore$ $A \propto r^2$

or $A = kr^2$ (i)

$\frac{1782}{7} = k(9)^2$

$\frac{1782}{7 \times 81} = k$ or $k = \frac{22}{7}$

Put $\quad k = \frac{22}{7}$ and r =14cm in eq. (i)

$$A = \frac{22}{7}(14)^2 = \frac{22}{7} \times 14 \times 14 = 616 \text{ cm}^2$$

Example 4: If y varies directly as cube of x and $y = 81$ when $x = 3$, so evaluate y when $x = 5$.

Solution: Given that y varies directly as cube of x.

i.e., $\quad y \propto x^3 \quad$ or $\quad y = kx^3 \quad$ (i) $\quad$ (where k is constant)

Put $y = 81$ and $x = 3$ in (i)

$$81 = k(3)^3 \Rightarrow 27k = 81 \Rightarrow k = 3$$

Now put $k = 3$ and $x = 5$ in eq. (i)

$$y = 3(5)^3 = 375$$

(ii) Inverse Variation

If two quantities are related in such a way that when one quantity increases, the other decreases is called **inverse variation**.

In otherwords, if a quantity y varies inversely with regard to quantity x. We say that y is inversely proportional to x or y varies inversely as x and is written as $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$.

i.e., $\quad xy = k$, where $k \neq 0$ is the constant of variation.

Example 1: If y varies inversely as x and $y = 8$, when $x = 4$. Find y, when $x = 16$.

Solution: Since y varies inversely as x, therefore

$$y \propto \frac{1}{x} \quad \text{or} \quad y = \frac{k}{x} \quad \text{(i)}$$

$$\Rightarrow \quad xy = k \quad \text{(ii)}$$

Putting $y = 8$ and $x = 4$ in (ii)

$$k = (x)(y)$$
$$= (4)(8) = 32$$

Now put $k = 32$ and $x = 16$ in (i) $\Rightarrow \quad y = \frac{32}{16} = 2$

Example 2: If y varies inversely as x^2 and $y = 16$, when $x = 5$, so find x, when $y = 100$.

Solution: Since y varies inversely as x^2, therefore

$$y \propto \frac{1}{x^2} \quad \text{or} \quad y = \frac{k}{x^2}$$

$$k = x^2 y \quad \text{(i)}$$

Put $x = 5$ and $y = 16$ in (i)

$$k = (5)^2 \times 16$$
$$k = 400$$

Now put $k = 400$ and $y = 100$ in (i)

$$400 = 100x^2 \quad \text{or} \quad x^2 = \frac{400}{100} = 4$$

$$x = \pm 2$$

EXERCISE 3.2

1. If y varies directly as x, and $y = 8$ when $x = 2$, find
   (i) y in terms of x (ii) y when $x = 5$
   (iii) x when $y = 28$
2. If $y \propto x$, and $y = 7$ when $x = 3$ find
   (i) y in terms of x (ii) x when $y = 35$ and y when $x = 18$
3. If $R \propto T$ and $R = 5$ when $T = 8$, find the equation connecting R and T. Also find R when $T = 64$ and T when $R = 20$.
4. If $R \propto T^2$ and $R = 8$ when $T = 3$, find R when $T = 6$.
5. If $V \propto R^3$ and $V = 5$ when $R = 3$, find R when $V = 625$.
6. If w varies directly as u^3 and $w = 81$ when $u = 3$. Find w when $u = 5$.
7. If y varies inversely as x and $y = 7$ when $x = 2$, find y when $x = 126$.
8. If $y \propto \frac{1}{x}$ and $y = 4$ when $x = 3$, find x when $y = 24$.
9. If $w \propto \frac{1}{z}$ and $w = 5$ when $z = 7$, find w when $z = \frac{175}{4}$.
10. $A \propto \frac{1}{r^2}$ and $A = 2$ when $r = 3$, find r when $A = 72$.
11. $a \propto \frac{1}{b^2}$ and $a = 3$ when $b = 4$, find a when $b = 8$.
12. $V \propto \frac{1}{r^3}$ and $V = 5$ when $r = 3$, find V when $r = 6$ and r when $V = 320$.
13. $m \propto \frac{1}{n^3}$ and $m = 2$ when $n = 4$, find m when $n = 6$ and n when $m = 432$.

3.1(ii) Find 3rd, 4th, mean and continued proportion:

We are already familiar with proportions that if quantities a, b, c and d are in proportion, then $a : b :: c : d$

i.e., product of extremes = product of means

Third Proportional

If three quantities a, b and c are related as $a : b :: b : c$,

then c is called the third proportion.

Example 1: Find a third proportional of $x + y$ and $x^2 - y^2$.

Solution: Let c be the third proportional,

then $x + y : x^2 - y^2 :: x^2 - y^2 : c$

$$c\,(x + y) = (x^2 - y^2)\,(x^2 - y^2)$$

$$c = \frac{(x^2 - y^2)(x^2 - y^2)}{x + y} = \frac{(x^2 - y^2)(x - y)(x + y)}{(x + y)}$$

$$c = (x^2 - y^2)(x - y) = (x + y)(x - y)^2$$

Fourth Proportional

If four quantities a, b, c and d are related as

$$a : b :: c : d$$

Then d is called the fourth proportional.

Example 2: Find fourth proportional of $a^3 - b^3$, $a + b$ and $a^2 + ab + b^2$

Solution: Let x be the fourth proportional,

then $(a^3 - b^3) : (a + b) :: (a^2 + ab + b^2) : x$

i.e., $x(a^3 - b^3) = (a + b)(a^2 + ab + b^2)$

$$x = \frac{(a + b)(a^2 + ab + b^2)}{a^3 - b^3} = \frac{(a + b)(a^2 + ab + b^2)}{(a - b)(a^2 + ab + b^2)}$$

$$x = \frac{a + b}{a - b}$$

Mean Proportional

If three quantities a, b and c are related as $a : b :: b : c$,

then b is called the mean proportional.

Example 3: Find the mean proportional of $9p^6q^4$ and r^8.

Solution: Let m be the mean proportional,

then $9p^6q^4 : m :: m : r^8$

or $m \,.\, m = 9p^6q^4\,(r^8)$

$$m^2 = 9p^6q^4r^8$$

$$m = \pm\sqrt{9p^6q^4r^8} = \pm\, 3p^3q^2r^4$$

Continued Proportion

If three quantities a, b and c are related as

$$a : b :: b : c \quad ,$$

where a is first, b is the mean and c is the third proportional, then a, b and c are in continued proportion.

Example 4: Find p, if 12, p and 3 are in continued proportion.

Solution: Since 12, p and 3 are in continued proportion.

$\therefore$ $12 : p :: p : 3$ *i.e.,* $p \,.\, p = (12)(3) \Rightarrow p^2 = 36$

Thus, $p = \pm\, 6$

EXERCISE 3.3

1. Find a third proportional to

(*i*)	6, 12	(*ii*)	$a^3, 3a^2$
(*iii*)	$a^2 - b^2, a - b$	(*iv*)	$(x - y)^2, x^3 - y^3$
(*v*)	$(x + y)^2, x^2 - xy - 2y^2$	(*vi*)	$\frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}$

2. Find a fourth proportional to

(*i*)	5, 8, 15	(*ii*)	$4x^4, 2x^3, 18x^5$
(*iii*)	$15a^5b^6, 10a^2b^5, 21a^3b^3$	(*iv*)	$x^2 - 11x + 24, (x - 3), 5x^4 - 40x^3$
(*v*)	$p^3 + q^3, p^2 - q^2, p^2 - pq + q^2$		
(*vi*)	$(p^2 - q^2)(p^2 + pq + q^2), p^3 + q^3, p^3 - q^3$		

3. Find a mean proportional between

(*i*)	20, 45	(*ii*)	$20x^3y^5, 5x^7y$
(*iii*)	$15p^4qr^3, 135q^5r^7$	(*iv*)	$x^2 - y^2, \frac{x - y}{x + y}$

4. Find the values of the letter involved in the following continued proportions.

(*i*)	5, p, 45	(*ii*)	8, x, 18
(*iii*)	12, $3p - 6$, 27	(*iv*)	7, $m - 3$, 28

3.2 Theorems on Proportions

If four quantities a, b, c and d form a proportion, then many other useful properties may be deduced by the properties of fractions.

(1) Theorem of Invertendo

If $a : b = c : d$, then $b : a = d : c$

Example 1: If $3m : 2n = p : 2q$, then

$2n : 3m = 2q : p$

Solution: Since $3m : 2n = p : 2q$

$$\therefore \quad \frac{3m}{2n} = \frac{p}{2q}$$

By invertendo theorem

$$\frac{2n}{3m} = \frac{2q}{p}$$

i.e., $2n : 3m = 2q : p$

(2) Theorem of Alternando

If $a : b = c : d$, then $a : c = b : d$

Example 2: If $3p + 1 : 2q = 5r : 7s$, then prove that $3p + 1 : 5r = 2q : 7s$

Solution: Given that $3p + 1 : 2q = 5r : 7s$

Then $\frac{3p+1}{2q} = \frac{5r}{7s}$

By alternando theorem

$$\frac{3p+1}{5r} = \frac{2q}{7s}$$

Thus, $3p + 1 : 5r = 2q : 7s$

(3) Theorem of Componendo

If $a : b = c : d$, then

(*i*) $a + b : b = c + d : d$

and (*ii*) $a : a + b = c : c + d$

Example 3: If $m + 3 : n = p : q - 2$, then

$$m + n + 3 : n = p + q - 2 : q - 2$$

Solution: Since $m + 3 : n = p : q - 2$

$\therefore \quad \frac{m+3}{n} = \frac{p}{q-2}$

By componendo theorem

$$\frac{(m+3)+n}{n} = \frac{p+(q-2)}{q-2}$$

or $\quad \frac{m+n+3}{n} = \frac{p+q-2}{q-2}$

Thus $\quad m + n + 3 : n = p + q - 2 : q - 2$

(4) Theorem of Dividendo

If $a : b = c : d$, then

(*i*) $a - b : b = c - d : d$

and (*ii*) $a : a - b = c : c - d$

Example 4: If $m + 1 : n - 2 = 2p + 3 : 3q + 1$.

Then $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

Solution: Given that $m + 1 : n - 2 = 2p + 3 : 3q + 1$

Then $\quad \frac{m+1}{n-2} = \frac{2p+3}{3q+1}$

By dividendo theorem

$$\frac{m-n+3}{n-2} = \frac{2p-3q+2}{3q+1}$$

Thus $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

(5) Theorem of Componendo-dividendo

If $a : b = c : d$, then

(i) $a + b : a - b = c + d : c - d$

and (ii) $a - b : a + b = c - d : c + d$

Example 5: If $m : n = p : q$.

Then prove that $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Solution: Since $m : n = p : q$

or $$\frac{m}{n} = \frac{p}{q}$$

Multiplying both sides by $\frac{3}{7}$, we get

$$\frac{3m}{7n} = \frac{3p}{7q}$$

Then using componendo-dividendo theorem

$$\frac{3m + 7n}{3m - 7n} = \frac{3p + 7q}{3p - 7q}$$

Thus $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Example 6: If $5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$,

then show that $m : n = p : q$

Solution: Given that $5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$

or $$\frac{5m + 3n}{5m - 3n} = \frac{5p + 3q}{5p - 3q}$$

By componendo-dividendo theorem

$$\frac{(5m + 3n) + (5m - 3n)}{(5m + 3n) - (5m - 3n)} = \frac{(5p + 3q) + (5p - 3q)}{(5p + 3q) - (5p - 3q)}$$

$$\frac{5m + 3n + 5m - 3n}{5m + 3n - 5m + 3n} = \frac{5p + 3q + 5p - 3q}{5p + 3q - 5p + 3q}$$

$$\frac{10m}{6n} = \frac{10p}{6q}$$

Multiplying both sides by $\frac{6}{10}$.

$$\frac{m}{n} = \frac{p}{q}$$

i.e., $m : n = p : q$

Example 7: Using theorem of componendo-dividendo, find the value of

$$\frac{m + 3p}{m - 3p} + \frac{m + 2q}{m - 2q}, \quad \text{if} \quad m = \frac{6pq}{p + q}.$$

Solution: Since $m = \dfrac{6pq}{p+q}$ or $m = \dfrac{(3p)(2q)}{p+q}$ (i)

$$\therefore \quad \frac{m}{3p} = \frac{2q}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+3p}{m-3p} = \frac{2q+(p+q)}{2q-(p+q)} = \frac{2q+p+q}{2q-p-q}$$

$$\frac{m+3p}{m-3p} = \frac{p+3q}{q-p} \qquad \text{(ii)}$$

Again from eq. (i), we have

$$\frac{m}{2q} = \frac{3p}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+2q}{m-2q} = \frac{3p+(p+q)}{3p-(p+q)} = \frac{3p+p+q}{3p-p-q}$$

$$\frac{m+2q}{m-2q} = \frac{4p+q}{2p-q} \qquad \text{(iii)}$$

Adding (ii) and (iii)

$$\frac{m+3p}{m-3p} + \frac{m+2q}{m-2q} = \frac{p+3q}{q-p} + \frac{4p+q}{2p-q} = -\frac{p+3q}{p-q} + \frac{4p+q}{2p-q}$$

$$= \frac{-(p+3q)(2p-q)+(p-q)(4p+q)}{(p-q)(2p-q)}$$

$$= \frac{-2p^2-5pq+3q^2+4p^2-3pq-q^2}{(p-q)(2p-q)}$$

$$= \frac{2p^2-8pq+2q^2}{(p-q)(2p-q)} = \frac{2(p^2-4pq+q^2)}{(p-q)(2p-q)}$$

Example 8: Using theorem of componendo-dividendo, solve the equation

$$\frac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}} = \frac{4}{3}.$$

Solution: Given equation is $\dfrac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}} = \dfrac{4}{3}$

By componendo-dividendo theorem

$$\frac{\sqrt{x+3}+\sqrt{x-3}+\sqrt{x+3}-\sqrt{x-3}}{\sqrt{x+3}+\sqrt{x-3}-\sqrt{x+3}+\sqrt{x-3}} = \frac{4+3}{4-3}$$

$$\frac{\not{2}\sqrt{x+3}}{\not{2}\sqrt{x-3}}=\frac{7}{1} \Rightarrow \sqrt{\frac{x+3}{x-3}}=7$$

Squaring both sides

$$\frac{x+3}{x-3}=49$$

$$x+3=49\,(x-3) \quad \Rightarrow x+3=49x-147 \quad \Rightarrow x-49x=-147-3$$

$$-48x=-150 \quad \Rightarrow 48x=150 \quad \Rightarrow x=\frac{150}{48}=\frac{25}{8}$$

Example 9: Using componendo-dividendo theorem, solve the equation $\frac{(x+3)^2-(x-5)^2}{(x+3)^2+(x-5)^2}=\frac{4}{5}$.

Solution: Given equation is $\frac{(x+3)^2-(x-5)^2}{(x+3)^2+(x-5)^2}=\frac{4}{5}$

By componendo-dividendo theorem

$$\frac{(x+3)^2-(x-5)^2+(x+3)^2+(x-5)^2}{(x+3)^2-(x-5)^2-(x+3)^2-(x-5)^2}=\frac{4+5}{4-5}$$

$$\frac{2(x+3)^2}{-2(x-5)^2}=\frac{9}{-1} \quad \Rightarrow \left(\frac{x+3}{x-5}\right)^2=(3)^2$$

Taking square root $\frac{x+3}{x-5}=\pm 3$

$$\frac{x+3}{x-5}=3 \qquad \text{or} \qquad \frac{x+3}{x-5}=-3$$

$x+3=3(x-5)$	$x+3=-3(x-5)$
$x+3=3x-15$	$x+3=-3x+15$
$-2x=-18$	$4x=12$
$x=9$	$x=3$

$\therefore$ The solution set is $\{3, 9\}$

EXERCISE 3.4

1. Prove that $a:b=c:d$, if

(i) $\frac{4a+5b}{4a-5b}=\frac{4c+5d}{4c-5d}$

(ii) $\frac{2a+9b}{2a-9b}=\frac{2c+9d}{2c-9d}$

(iii) $\frac{ac^2+bd^2}{ac^2-bd^2}=\frac{c^3+d^3}{c^3-d^3}$

(iv) $\frac{a^2c+b^2d}{a^2c-b^2d}=\frac{ac^2+bd^2}{ac^2-bd^2}$

(v) $pa+qb:pa-qb=pc+qd:pc-qd$

(vi) $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$

(vii) $\dfrac{2a+3b+2c+3d}{2a+3b-2c-3d}=\dfrac{2a-3b+2c-3d}{2a-3b-2c+3d}$

(viii) $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{ac+bd}{ac-bd}$

2. Using theorem of componendo-dividendo

(i) Find the value of $\dfrac{x+2y}{x-2y}+\dfrac{x+2z}{x-2z}$, if $x=\dfrac{4yz}{y+z}$

(ii) Find the value of $\dfrac{m+5n}{m-5n}+\dfrac{m+5p}{m-5p}$, if $m=\dfrac{10np}{n+p}$

(iii) Find the value of $\dfrac{x-6a}{x+6a}-\dfrac{x+6b}{x-6b}$, if $x=\dfrac{12ab}{a-b}$

(iv) Find the value of $\dfrac{x-3y}{x+3y}-\dfrac{x+3z}{x-3z}$, if $x=\dfrac{3yz}{y-z}$

(v) Find the value of $\dfrac{s-3p}{s+3p}+\dfrac{s+3q}{s-3q}$, if $s=\dfrac{6pq}{p-q}$

(vi) Solve $\dfrac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2}=\dfrac{12}{13}$

(vii) Solve $\dfrac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}}=2$

(viii) Solve $\dfrac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}=\dfrac{1}{3}$

(ix) Solve $\dfrac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3}=\dfrac{13}{14}$

3.3.(i) Joint variation

A combination of direct and inverse variations of one or more than one variables forms **joint variation**.

If a variable y varies directly as x and varies inversely as z.

Then $y \propto x$ and $y \propto \dfrac{1}{z}$

In joint variation, we write it as

$$y \propto \frac{x}{z}$$

i.e., $y = k\dfrac{x}{z}$

Where $k \neq 0$ is the constant of variation.

For example, by Newton's law of gravitation, if one body attracts another with a force (G), that varies directly as the product of their masses (m_1), (m_2) and inversely as the square of the distance (d) between them.

i.e., $G \propto \dfrac{m_1 m_2}{d^2}$ or $G = k\dfrac{m_1 m_2}{d^2}$, where $k \neq 0$ is the constant

3.3.(*ii*) Problems related to joint variation.

Procedure to solve the problems related to joint variation is explained through examples.

Example 1: If y varies jointly as x^2 and z and $y = 6$ when $x = 4$, $z = 9$. Write y as a function of x and z and determine the value of y, when $x = -8$ and $z = 12$.

Solution: Since y varies jointly as x^2 and z, therefore

$$y \propto x^2 z$$

i.e., $y = kx^2 z$ (i)

Put $y = 6, x = 4, z = 9$

$$6 = k(4)^2(9)$$

$$\frac{6}{16 \times 9} = k \quad \Rightarrow \quad k = \frac{1}{24}$$

Put $k = \dfrac{1}{24}$ in eq.(i), $y = \dfrac{1}{24}x^2 z$

Now put $x = -8$, $z = 12$ in the above equation,

$$y = \frac{1}{24}(-8)^2(12) = 32$$

Example 2: p varies jointly as q and r^2 and inversely as s and t^2, $p = 40$, when $q = 8$, $r = 5$, $s = 3$, $t = 2$. Find p in terms of q, r, s and t. Also find the value of p when $q = -2$, $r = 4$, $s = 3$ and $t = -1$.

Solution: Given that $p \propto \dfrac{qr^2}{st^2}$

$$p = k\frac{qr^2}{st^2} \qquad \text{(i)}$$

Put $p = 40$, $q = 8$, $r = 5$, $s = 3$ and $t = 2$

$$40 = k\frac{(8)(5)^2}{3(2)^2}$$

$$\frac{40 \times 3 \times 4}{8 \times 25} = k$$

$$k = \frac{12}{5}$$

Then eq. (i) becomes

$$p = \frac{12}{5}\,\frac{qr^2}{st^2}$$

Now for $q = -2$, $r = 4$, $s = 3$ and $t = -1$, we have

$$p = \frac{12}{5}\ \frac{(-2)\,(4)^2}{(3)\,(-1)^2} = -\frac{128}{5}$$

EXERCISE 3.5

1. If s varies directly as u^2 and inversely as v and $s = 7$ when $u = 3$, $v = 2$. Find the value of s when $u = 6$ and $v = 10$.
2. If w varies jointly as x, y^2 and z and $w = 5$ when $x = 2$, $y = 3$, $z = 10$. Find w when $x = 4$, $y = 7$ and $z = 3$.
3. If y varies directly as x^3 and inversely as z^2 and t, and $y = 16$ when $x = 4$, $z = 2$, $t = 3$. Find the value of y when $x = 2$, $z = 3$ and $t = 4$.
4. If u varies directly as x^2 and inversely as the product yz^3, and $u = 2$ when $x = 8$, $y = 7$, $z = 2$. Find the value of u when $x = 6$, $y = 3$, $z = 2$.
5. If v varies directly as the product xy^3 and inversely as z^2 and $v = 27$ when $x = 7$, $y = 6$, $z = 7$. Find the value of v when $x = 6$, $y = 2$, $z = 3$.
6. If w varies inversely as the cube of u, and $w = 5$ when $u = 3$. Find w when $u = 6$.

3.4. *K*-Method

3.4(i) Use *k*-method to prove conditional equalities involving proportions.

If $a : b :: c : d$ is a proportion, then putting each ratio equal to k

i.e., $\frac{a}{b} = \frac{c}{d} = k$

$\frac{a}{b} = k$ and $\frac{c}{d} = k$

$a = bk$ and $c = dk$

Using the above equations, we can solve certain problems relating to proportions more easily. This method is known as *k*-method. We illustrate the *k*-method through the following examples.

Example 1: If $a : b = c : d$, then show that

$$\frac{3a + 2b}{3a - 2b} = \frac{3c + 2d}{3c - 2d}$$

Solution: $a : b = c : d$

Let $\frac{a}{b} = \frac{c}{d} = k$

Then $a = bk$ and $c = dk$

To prove $\frac{3a + 2b}{3a - 2b} = \frac{3c + 2d}{3c - 2d}$

Now L.H.S $= \frac{3a+2b}{3a-2b} = \frac{3kb+2b}{3kb-2b} = \frac{b(3k+2)}{b(3k-2)}$

$= \frac{3k+2}{3k-2}$ (i)

Also R.H.S $= \frac{3c+2d}{3c-2d} = \frac{3kd+2d}{3kd-2d} = \frac{d(3k+2)}{d(3k-2)}$

$= \frac{3k+2}{3k-2}$ (ii)

$\therefore$ L.H.S = R.H.S

i.e., $\frac{3a+2b}{3a-2b} = \frac{3c+2d}{3c-2d}$

Example 2: If $a : b = c : d$, then show that

$pa + qb : ma - nb = pc + qd : mc - nd$

Solution: Let $\frac{a}{b} = \frac{c}{d} = k$, then $a = bk$ and $c = dk$

L.H.S $= pa + qb : ma - nb = \frac{pa+qb}{ma-nb} = \frac{pkb+qb}{mkb-nb}$

$= \frac{b(pk+q)}{b(mk-n)} = \frac{pk+q}{mk-n}$

R.H.S $= pc + qd : mc - nd = \frac{pc+qd}{mc-nd} = \frac{pkd+qd}{mkd-nd}$ $(c = kd)$

$= \frac{d(pk+q)}{d(mk-n)} = \frac{pk+q}{mk-n}$

i.e., $pa + qb : ma - nb = pc + qd : mc - nd$

Example 3: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then show that $\frac{a^3+c^3+e^3}{b^3+d^3+f^3} = \frac{ace}{bdf}$

Solution: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

Then $\frac{a}{b} = k$, $\frac{c}{d} = k$ and $\frac{e}{f} = k$

i.e., $a = bk$, $c = dk$ and $e = fk$

To prove $\frac{a^3+c^3+e^3}{b^3+d^3+f^3} = \frac{ace}{bdf}$

Now L.H.S $= \frac{a^3+c^3+e^3}{a^3+d^3+f^3} = \frac{(bk)^3+(dk)^3+(fk)^3}{b^3+d^3+f^3}$

$= \frac{b^3k^3+d^3k^3+f^3k^3}{b^3+d^3+f^3} = k^3\left(\frac{b^3+d^3+f^3}{b^3+d^3+f^3}\right) = k^3$

Also R.H.S $= \frac{ace}{bdf} = \frac{(bk)\,(dk)\,(fk)}{bdf} = k^3\,\frac{bdf}{bdf} = k^3$

$\therefore$ L.H.S = R.H.S

i.e., $\frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \frac{ace}{bdf}$

Example 4: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then show that $\frac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2} = \frac{a + c + e}{b + d + f}$

Solution: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$a = bk,\ c = dk,\ e = fk$

To prove $\frac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2} = \frac{a + c + e}{b + d + f}$

L.H.S. $= \frac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2}$

$= \frac{(bk)^2b + (dk)^2d + (fk)^2f}{(bk)b^2 + (dk)d^2 + (fk)f^2} = \frac{k^2b^3 + k^2d^3 + k^2f^3}{kb^3 + kd^3 + kf^3}$

$= \frac{k^2\,(b^3 + d^3 + f^3)}{k\,(b^3 + d^3 + f^3)} = k$

R.H.S. $= \frac{a + c + e}{b + d + f} = \frac{bk + dk + fk}{b + d + f}$

$= \frac{k\,(b + d + f)}{b + d + f} = k$

L.H.S. = R.H.S.

Thus, $\frac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2} = \frac{a + c + e}{b + d + f}$

EXERCISE 3.6

1. If $a : b = c : d$, $(a, b, c, d \neq 0)$, then show that

(i) $\frac{4a - 9b}{4a + 9b} = \frac{4c - 9d}{4c + 9d}$ (ii) $\frac{6a - 5b}{6a + 5b} = \frac{6c - 5d}{6c + 5d}$

(iii) $\frac{a}{b} = \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$ (iv) $a^6 + c^6 : b^6 + d^6 = a^3c^3 : b^3d^3$

(v) $p(a + b) + qb : p(c + d) + qd = a : c$

(vi) $a^2 + b^2 : \frac{a^3}{a + b} = c^2 + d^2 : \frac{c^3}{c + d}$

(vii) $\frac{a}{a - b} : \frac{a + b}{b} = \frac{c}{c - d} : \frac{c + d}{d}$

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ $(a, b, c, d, e, f \neq 0)$, then show that

(i) $\frac{a}{b}=\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$ (ii) $\frac{ac+ce+ea}{bd+df+fb}=\left[\frac{ace}{bdf}\right]^{2/3}$

(iii) $\frac{ac}{bd}+\frac{ce}{df}+\frac{ea}{fb}=\frac{a^2}{b^2}+\frac{c^2}{d^2}+\frac{e^2}{f^2}$

3.4(ii) Real life problems based on variation

Example 1: The strength "s" of a rectangular beam varies directly as the breadth b and the square of the depth d. If a beam 9cm wide and 12cm deep will support 1200 lb. What weight a beam of 12cm wide and 9cm deep will support?

Solution: By the joint variation, we have $s \propto bd^2$

i.e., $s = kbd^2$ (i)

Put $s = 1200$, $b = 9$ and $d = 12$

$k(9)(12)^2 = 1200$

$k = \frac{1200}{9\times 144}=\frac{25}{27}$

Put in eq. (i) $s=\frac{25}{27}bd^2$

Now for $b = 12$ and $d = 9$

$s=\frac{25}{27}(12)(9)^2 = \frac{25(12)(9)(9)}{27} = 900$ lb

Example 2: The current in a wire varies directly as the electromotive force E and inversely as the resistance R. If $I = 32$ amperes, when $E = 128$ volts and $R = 8$ ohms. Find I, when $E = 150$ volts and $R = 18$ ohms.

Solution: In joint variation, we have $I \propto \frac{E}{R}$, *i.e.,* $I=\frac{kE}{R}$ (i)

For $I = 32$, $E = 128$ and $R = 8$,

$32=\frac{k(128)}{8} \Rightarrow \frac{32\times 8}{128}=k \Rightarrow k=2$

Put in eq. (i) $I=\frac{2E}{R}$.

Now for $E = 150$ and $R = 18$

$I=\frac{2(150)}{18}=\frac{50}{3}$ amp.

EXERCISE 3.7

1. The surface area A of a cube varies directly as the square of the length l of an edge and $A = 27$ square units when $l = 3$ units.

   Find (i) A when $l = 4$ units (ii) l when $A = 12$ sq. units.

2. The surface area S of the sphere varies directly as the square of radius r,

   and $S = 16\pi$ when $r = 2$. Find r when $S = 36\pi$.

3. In Hook's law the force F applied to stretch a spring varies directly as the amount of elongation S and $F = 32lb$ when $S = 1.6$ in. Find (i) S when $F = 50$ lb (ii) F when $S = 0.8$in.

4. The intensity I of light from a given source varies inversely as the square of the distance d from it. If the intensity is 20 candlepower at a distance of 12ft. from the source, find the intensity at a point 8ft. from the source.

5. The pressure P in a body of fluid varies directly as the depth d. If the pressure exerted on the bottom of a tank by a column of fluid 5ft. high is 2.25 lb/sq. in, how deep must the fluid be to exert a pressure of 9 lb/sq. in?

6. Labour costs c varies jointly as the number of workers n and the average number of days d. If the cost of 800 workers for 13 days is Rs. 286000, then find the labour cost of 600 workers for 18 days.

7. The supporting load c of a pillar varies as the fourth power of its diameter d and inversely as the square of its length l. A pillar of diameter 6 inch and of height 30 feet will support a load of 63 tons. How high a 4 inch pillar must be to support a load of 28 tons?

8. The time T required for an elevator to lift a weight varies jointly as the weight w and the lifting depth d varies inversely as the power p of the motor. If 25 sec. are required for a 4-hp motor to lift 500 lb through 40 ft, what power is required to lift 800 lb, through 120 ft in 40 sec.?

9. The kinetic energy (K.E.) of a body varies jointly as the mass "m" of the body and the square of its velocity "v". If the kinetic energy is 4320 ft/lb when the mass is 45 lb and the velocity is 24 ft/sec, determine the kinetic energy of a 3000 lb automobile travelling 44 ft/sec.

MISCELLANEOUS EXERCISE - 3

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) In a ratio $a : b$, a is called

(a) relation	(b) antecedent
(c) consequent	(d) None of these

(ii) In a ratio $x : y$, y is called

(a) relation (b) antecedent

(c) consequent (d) None of these

(iii) In a proportion $a : b : : c : d$, a and d are called,

(a) means (b) extremes

(c) third proportional (d) None of these

(iv) In a proportion $a : b : : c : d$, b and c are called

(a) means (b) extremes

(c) fourth proportional (d) None of these

(v) In continued proportion $a : b = b : c$, $ac = b^2$, b is said to be _______ proportional between a and c.

(a) third (b) fourth

(c) means (d) None of these

(vi) In continued proportion $a : b = b : c$, c is said to be ______ proportional to a and b.

(a) third (b) fourth

(c) means (d) None of these

(vii) Find x in proportion $4 : x : : 5 : 15$

(a) $\frac{75}{4}$ (b) $\frac{4}{3}$

(c) $\frac{3}{4}$ (d) 12

(viii) If $u \propto v^2$, then

(a) $u = v^2$ (b) $u = kv^2$

(c) $uv^2 = k$ (d) $uv^2 = 1$

(ix) If $y^2 \propto \frac{1}{x^3}$, then

(a) $y^2 = \frac{k}{x^3}$ (b) $y^2 = \frac{1}{x^3}$

(c) $y^2 = x^2$ (d) $y^2 = kx^3$

(x) If $\frac{u}{v} = \frac{v}{w} = k$, then

(a) $u = wk^2$ (b) $u = vk^2$

(c) $u = w^2k$ (d) $u = v^2k$

(xi) The third proportional of x^2 and y^2 is

(a) $\frac{y^2}{x^2}$ (b) x^2y^2

(c) $\frac{y^4}{x^2}$ (d) $\frac{y^2}{x^4}$

(xii) The fourth proportional w of $x : y :: v : w$ is

(a) $\frac{xy}{v}$ (b) $\frac{vy}{x}$

(c) xyv (d) $\frac{x}{vy}$

(xiii) If $a : b = x : y$, then alternando property is

(a) $\frac{a}{x} = \frac{b}{y}$ (b) $\frac{a}{b} = \frac{x}{y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$ (d) $\frac{a-b}{x} = \frac{x-y}{y}$

(xiv) If $a : b = x : y$, then invertendo property is

(a) $\frac{a}{x} = \frac{b}{y}$ (b) $\frac{a}{a-b} = \frac{x}{x-y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$ (d) $\frac{b}{a} = \frac{y}{x}$

(xv) If $\frac{a}{b} = \frac{c}{d}$, then componendo property is

(a) $\frac{a}{a+b} = \frac{c}{c+d}$ (b) $\frac{a}{a-b} = \frac{c}{c-d}$

(c) $\frac{ad}{bc}$ (d) $\frac{a-b}{b} = \frac{c-d}{d}$

2. Write short answers of the following questions.

(i) Define ratio and give one example.

(ii) Define proportion.

(iii) Define direct variation.

(iv) Define inverse variation.

(v) State theorem of componendo-dividendo.

(vi) Find x, if $6 : x :: 3 : 5$.

(vii) If x and y^2 varies directly, and $x = 27$ when $y = 4$. Find the value of y when $x = 3$.

(viii) If u and v varies inversely, and $u = 8$, when $v = 3$. Find v when $u = 12$.

(ix) Find the fourth proportional to 8, 7, 6.

(x) Find a mean proportional to 16 and 49.

(xi) Find a third proportional to 28 and 4.

(xii) If $y \propto \frac{x^2}{z}$ and $y = 28$ when $x = 7$, $z = 2$, then find y.

(xiii) If $z \propto xy$ and $z = 36$ when $x = 2$, $y = 3$, then find z.

(xiv) If $w \propto \frac{1}{v^2}$ and $w = 2$ when $v = 3$, then find w.

3. Fill in the blanks

(i) The simplest form of the ratio $\frac{(x+y)(x^2+xy+y^2)}{x^3-y^3}$ is ________.

(ii) In a ratio $x : y$; x is called ________.

(iii) In a ratio $a : b$; b is called ________.

(iv) In a proportion $a : b :: x : y$; a and y are called ________.

(v) In a proportion $p : q :: m : n$; q and m are called ________.

(vi) In proportion $7 : 4 :: p : 8$, p = ________.

(vii) If $6 : m :: 9 : 12$, then m = ________.

(viii) If x and y varies directly, then x = ________.

(ix) If v varies directly as u^3, then u^3 = ________.

(x) If w varies inversely as p^2, then k = ________.

(xi) A third proportional of 12 and 4, is ________.

(xii) The fourth proportional of 15, 6, 5 is ________.

(xiii) The mean proportional of $4m^2n^4$ and p^6 is ________.

(xiv) The continued proportion of 4, m and 9 is ________.

SUMMARY

- A relation between two quantities of the same kind is called **ratio**.
- A **proportion** is a statement, which is expressed as equivalence of two ratios.

  If two ratios $a : b$ and $c : d$ are equal, then we can write $a : b = c : d$

- If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) in the other quantity is called **direct variation**.
- If two quantities are related in such a way that when one quantity increases, the other decreases is called **inverse variation**.
- Theorem on proportions:

  (1) **Theorem of Invertendo**

  If $a : b = c : d$, then $b : a = d : c$

  (2) **Theorem of Alternando**

  If $a : b = c : d$, then $a : c = b : d$

  (3) **Theorem of Componendo**

  If $a : b = c : d$, then

  (*i*) $a + b : b = c + d : d$

  and (*ii*) $a : a + b = c : c + d$

(4) **Theorem of Dividendo**

If $a : b = c : d$, then

(*i*) $a - b : b = c - d : d$

(*ii*) $a : a - b = c : c - d$

(5) **Theorem of Componendo-dividendo**

If $a : b = c : d$, then

$a + b : a - b = c + d : c - d$

- A combination of direct and inverse variations of one or more than one variable forms **joint variation**.
- ***K*-Method,**

(a) If $\frac{a}{b} = \frac{c}{d}$,

then $\frac{a}{b} = \frac{c}{d} = k$ or $a = bk$ and $c = dk$

(b) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$, then $a = bk$, $c = dk$ and $e = fk$