Unit-5

SETS AND FUNCTIONS

In this unit, students will learn how to

- *sets*
- *recall the sets denoted by N, W, Z, E, O, P and Q.*
- *recognize operation on sets (∪, ∩, \, ...)*
- *perform operations on sets union, intersection, difference, complement.*
- *give formal proofs of the following fundamental properties of union and intersection of two or three sets.*
  - *commutative property of union,*
  - *commutative property of intersection,*
  - *associative property of union,*
  - *associative property of intersection,*
  - *distributive property of union over intersection,*
  - *distributive property of intersection over union,*
  - *De Morgan's laws.*
- *verify the fundamental properties for given sets.*
- *use Venn diagram to represent*
  - *union and intersection of sets,*
  - *complement of a set.*
- *use Venn diagram to verify*
  - *commutative law for union and intersection of sets,*
  - *De Morgan's laws,*
  - *associative laws,*
  - *distributive laws.*
- *recognize ordered pairs and cartesian product.*
- *define binary relation and identify its domain and range.*
- *define function and identify its domain, co-domain and range.*
- *demonstrate the following*
  - *into function,*
  - *one-one function,*
  - *into and one-one function (injective function),*
  - *onto function (surjective function),*
  - *one-one and onto function (bijective function).*
- *examine whether a given relation is a function or not.*
- *differentiate between one-one correspondence and one-one function.*
- *include sufficient exercises to classify/differentiate between the above concepts.*

5.1 SETS

A set is a well-defined collection of objects and it is denoted by capital letters A, B, C etc.

5.1.1(i) Some Important Sets:

In set theory, we usually deal with the following sets of numbers denoted by standard symbols:

N = The set of natural numbers = $\{1, 2, 3, 4, \cdots\}$

W = The set of whole numbers = $\{0, 1, 2, 3, 4, \cdots\}$

Z = The set of all integers = $\{0, \pm 1, \pm 2, \pm 3, \cdots\}$

E = The set of all even integers = $\{0, \pm 2, \pm 4, \cdots\}$

O = The set of all odd integers = $\{\pm 1, \pm 3, \pm 5, \cdots\}$

P = The set of prime numbers = $\{2, 3, 5, 7, 11, 13, 17, \cdots\}$

Q = The set of all rational numbers = $\{x \mid x = \frac{m}{n}$, where $m, n \in Z$ and $n \neq 0\}$

Q' = The set of all irrational numbers = $\{x \mid x \neq \frac{m}{n}$, where $m, n \in Z$ and $n \neq 0\}$

R = The set of all real numbers = $Q \cup Q'$.

5.1.1(ii) Recognize operations on sets ($\cup$, $\cap$, \, ...):

(a) Union of sets

The union of two sets A and B written as $A \cup B$ (read as A union B) is the set consisting of all the elements which are either in A or in B or in both. Thus

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \text{ and } B \text{ both}\}.$$

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

(b) Intersection of sets

The intersection of two sets A and B, written as $A \cap B$ (read as 'A intersection B') is the set consisting of all the common elements of A and B. Thus

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Clearly $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$, then

$$A \cap B = \{c, d\}$$

(c) Difference of sets

If A and B are two sets, then their difference $A - B$ or $A \setminus B$ is defined as:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Similarly $\quad B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5, 6, 8\}$, then

$$A - B = \{1, 2, 3, 4, 5\} - \{2, 4, 5, 6, 8\} = \{1, 3\}$$

Also $\quad B - A = \{2, 4, 5, 6, 8\} - \{1, 2, 3, 4, 5\} = \{6, 8\}$.

(d) Complement of a set

If U is a universal set and A is a subset of U, then the complement of A is the set of those elements of U, which are not contained in A and is denoted by A' or A^c.

$\therefore$ $A' = U - A = \{x | x \in U \text{ and } x \notin A\}$.

For example, if $U = \{1, 2, 3, ..., 10\}$ and $A = \{2, 4, 6, 8\}$, then

$A' = U - A$

$= \{1, 2, 3, ..., 10\} - \{2, 4, 6, 8\}$

$= \{1, 3, 5, 7, 9, 10\}$

5.1.1(iii) Perform operations on sets:

Example: If $U = \{1, 2, 3, ..., 10\}$, $A = \{2, 3, 5, 7\}$, $B = \{3, 5, 8\}$, then

find (i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$

(iv) A' and B'

Solution: (i) $A \cup B = \{2, 3, 5, 7\} \cup \{3, 5, 8\}$

$= \{2, 3, 5, 7, 8\}$

(ii) $A \cap B = \{2, 3, 5, 7\} \cap \{3, 5, 8\}$

$= \{3, 5\}$

(iii) $A \backslash B = \{2, 3, 5, 7\} \backslash \{3, 5, 8\}$

$= \{2, 7\}$

(iv) $A' = U - A = \{1, 2, 3, ..., 10\} - \{2, 3, 5, 7\}$

$= \{1, 4, 6, 8, 9, 10\}$

$B' = U - B = \{1, 2, 3, ..., 10\} - \{3, 5, 8\}$

$= \{1, 2, 4, 6, 7, 9, 10\}$

EXERCISE 5.1

1. If $X = \{1, 4, 7, 9\}$ and $Y = \{2, 4, 5, 9\}$

   Then find:

   (i) $X \cup Y$ (ii) $X \cap Y$

   (iii) $Y \cup X$ (iv) $Y \cap X$

2. If X = Set of prime numbers less than or equal to 17

   and Y = Set of first 12 natural numbers, then find the following

   (i) $X \cup Y$ (ii) $Y \cup X$ (iii) $X \cap Y$ (iv) $Y \cap X$

3. If $X = \phi$, $Y = Z^+$, $T = O^+$, then

   find: (i) $X \cup Y$ (ii) $X \cup T$ (iii) $Y \cup T$

   (iv) $X \cap Y$ (v) $X \cap T$ (vi) $Y \cap T$

4. If $U = \{x | x \in N \wedge 3 < x \leq 25\}$, $X = \{x | x \text{ is prime} \wedge 8 < x < 25\}$

   and $Y = \{x | x \in W \wedge 4 \leq x \leq 17\}$.

Find the value of:

(i) $(X \cup Y)'$ (ii) $X' \cap Y'$

(iii) $(X \cap Y)'$ (iv) $X' \cup Y'$

5. If $X = \{2, 4, 6, \cdots, 20\}$ and $Y = \{4, 8, 12, \cdots, 24\}$, then find the following:

(i) $X - Y$ (ii) $Y - X$

6. If $A = N$ and $B = W$, then find the value of

(i) $A - B$ (ii) $B - A$

5.1.2(iv) Properties of Union and Intersection:

(a) Commutative property of union.

For any two sets A and B, prove that $A \cup B = B \cup A$.

Proof:

Let $x \in A \cup B$

$\Rightarrow$ $x \in A$ or $x \in B$ (by definition of union of sets)

$\Rightarrow$ $x \in B$ or $x \in A$

$\Rightarrow$ $x \in B \cup A$

$\Rightarrow$ $A \cup B \subseteq B \cup A$ (i)

Now let $y \in B \cup A$

$\Rightarrow$ $y \in B$ or $y \in A$ (by definition of union of sets)

$\Rightarrow$ $y \in A$ or $y \in B$

$\Rightarrow$ $y \in A \cup B$

$\Rightarrow$ $B \cup A \subseteq A \cup B$ (ii)

From (i) and (ii), we have $A \cup B = B \cup A$. (by definition of equal sets)

(b) Commutative property of intersection

For any two sets A and B, prove that $A \cap B = B \cap A$

Proof: Let $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$ (by definition of intersection of sets)

$\Rightarrow$ $x \in B$ and $x \in A$

$\Rightarrow$ $x \in B \cap A$

$\therefore$ $A \cap B \subseteq B \cap A$ (i)

Now let $y \in B \cap A$

$\Rightarrow$ $y \in B$ and $y \in A$ (by definition of intersection of sets)

$\Rightarrow$ $y \in A$ and $y \in B$

$\Rightarrow$ $y \in A \cap B$

Therefore, $B \cap A \subseteq A \cap B$ (ii)

From (i) and (ii), we have $A \cap B = B \cap A$ (by definition of equal sets)

(c) Associative property of union

For any three sets A, B and C, prove that $(A \cup B) \cup C = A \cup (B \cup C)$

Proof: Let $x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B \cup C$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\Rightarrow$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ (i)

Similarly $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ (ii)

From (i) and (ii), we have

$$(A \cup B) \cup C = A \cup (B \cup C)$$

(d) Associative property of intersection

For any three sets A, B and C, prove that $(A \cap B) \cap C = A \cap (B \cap C)$

Proof: Let $x \in (A \cap B) \cap C$

$\Rightarrow$ $x \in (A \cap B)$ and $x \in C$

$\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow$ $x \in A$ and $(x \in B$ and $\in C)$

$\Rightarrow$ $x \in A$ and $x \in B \cap C$

$\Rightarrow$ $x \in A \cap (B \cap C)$

$\therefore$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ (i)

Similarly $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ (ii)

From (i) and (ii), we have

$$(A \cap B) \cap C = A \cap (B \cap C)$$

(e) Distributive property of union over intersection

For any three sets A, B and C, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: Let $x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (i)

Similarly, now let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow$ $y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow$ $(y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow$ $y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow$ $y \in A$ or $y \in B \cap C$

$\Rightarrow$ $y \in A \cup (B \cap C)$

$\Rightarrow$ $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (ii)

From (i) and (ii), we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) Distributive property of intersection over union

For any three sets A, B and C, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Let $x \in A \cap (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \in B \cup C$

$\Rightarrow$ $x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow$ $(x \in A \cap B)$ or $(x \in A \cap C)$

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ (i)

Similarly $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ (ii)

From (i) and (ii), we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(g) De-Morgan's laws

For any two sets A and B

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Proof: Let $x \in (A \cup B)'$

$\Rightarrow$ $x \notin A \cup B$ (by definition of complement of set)

$\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A'$ and $x \in B'$

$\Rightarrow$ $x \in A' \cap B'$ (by definition of intersection of sets)

$\Rightarrow$ $(A \cup B)' \subseteq A' \cap B'$ (i)

Similarly $A' \cap B' \subseteq (A \cup B)'$ (ii)

Using (i) and (ii), we have $(A \cup B)' = A' \cap B'$

(ii) Let $x \in (A \cap B)'$

$\Rightarrow$ $x \notin A \cap B$

$\Rightarrow$ $x \notin A$ or $x \notin B$

$\Rightarrow$ $x \in A'$ or $x \in B'$

$\Rightarrow$ $x \in A' \cup B'$

$\Rightarrow$ $(A \cap B)' \subseteq A' \cup B'$ (i)

Let $y \in A' \cup B'$

$\Rightarrow$ $y \in A'$ or $y \in B'$

$\Rightarrow$ $y \notin A$ or $y \notin B$

$\Rightarrow$ $y \notin A \cap B$

$\Rightarrow$ $y \in (A \cap B)'$

$\Rightarrow$ $A' \cup B' \subseteq (A \cap B)'$ (ii)

From (i) and (ii), we have proved that

$(A \cap B)' = A' \cup B'$

EXERCISE 5.2

1. If $X = \{1, 3, 5, 7, \cdots, 19\}$, $Y = \{0, 2, 4, 6, 8, \cdots, 20\}$
and $Z = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$,
then find the following:
(i) $X \cup (Y \cup Z)$ (ii) $(X \cup Y) \cup Z$
(iii) $X \cap (Y \cap Z)$ (iv) $(X \cap Y) \cap Z$
(v) $X \cup (Y \cap Z)$ (vi) $(X \cup Y) \cap (X \cup Z)$
(vii) $X \cap (Y \cup Z)$ (viii) $(X \cap Y) \cup (X \cap Z)$
2. If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 4, 8\}$.
Prove the following identities:
(i) $A \cap B = B \cap A$ (ii) $A \cup B = B \cup A$
(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{1, 3, 5, 7, 9\}$, $B = \{2, 3, 5, 7\}$,
then verify the De-Morgan's Laws
i.e., $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$
4. If $U = \{1, 2, 3, \cdots, 20)$, $X = \{1, 3, 7, 9, 15, 18, 20\}$
and $Y = \{1, 3, 5, \cdots, 17\}$, then show that
(i) $X - Y = X \cap Y'$ (ii) $Y - X = Y \cap X'$

5.1.2(v) Verify the fundamental properties for given sets:

(a) A and B are any two subsets of U, then $A \cup B = B \cup A$ (commutative law).

For example $A = \{1, 3, 5, 7\}$ and $B = \{2, 3, 5, 7\}$
then $A \cup B = \{1, 3, 5, 7\} \cup \{2, 3, 5, 7\} = \{1, 2, 3, 5, 7\}$
and $B \cup A = \{2, 3, 5, 7\} \cup \{1, 3, 5, 7\} = \{1, 2, 3, 5, 7\}$
Hence, verified that $A \cup B = B \cup A$.

(b) Commutative property of intersection

For example $A = \{1, 3, 5, 7\}$ and $B = [2, 3, 5, 7\}$
Then $A \cap B = \{1, 3, 5, 7\} \cap \{2, 3, 5, 7\} = \{3, 5, 7\}$
and $B \cap A = \{2, 3, 5, 7\} \cap \{1, 3, 5, 7\} = \{3, 5, 7\}$
Hence, verified that $A \cap B = B \cap A$.

(c) If A, B and C are the subsets of U, then $(A \cup B) \cup C = A \cup (B \cup C)$.

(Associative law)
Suppose $A = \{1, 2, 4, 8\}$; $B = \{2, 4, 6\}$
and $C = \{3, 4, 5, 6\}$

then L.H.S $= (A \cup B) \cup C$

$= (\{1, 2, 4, 8\} \cup \{2, 4, 6\}) \cup \{3, 4, 5, 6\}$

$= \{1, 2, 4, 6, 8\} \cup \{3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

and R.H.S. $= A \cup (B \cup C)$

$= \{1, 2, 4, 8\} \cup (\{2, 4, 6\} \cup \{3, 4, 5, 6\})$

$= \{1, 2, 4, 8\} \cup \{2, 3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

L.H.S. = R.H.S.

Hence, union of sets is associative.

(d) If A, B and C are the subsets of U, then $(A \cap B) \cap C = A \cap (B \cap C)$

(Associative Law).

Suppose $A = \{1, 2, 4, 8\}$; $B = \{2, 4, 6\}$ and $C = \{3, 4, 5, 6\}$

then L.H.S. $= (A \cap B) \cap C$

$= (\{1, 2, 4, 8\} \cap \{2, 4, 6\}) \cap \{3, 4, 5, 6\}$

$= \{2, 4\} \cap \{3, 4, 5, 6\} = \{4\}$

and R.H.S. $= A \cap (B \cap C)$

$= \{1, 2, 4, 8\} \cap (\{2, 4, 6\} \cap \{3, 4, 5, 6\})$

$= \{1, 2, 4, 8\} \cap \{4, 6\} = \{4\}$

L.H.S. = R.H.S.

Hence, intersection of sets is associative.

Distributive laws

(e) Union is distributive over intersection of sets

If A, B and C are the subsets of universal set U, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution: Suppose $A = \{1, 2, 4, 8\}$, $B = \{2, 4, 6\}$ and $C = \{3, 4, 5, 6\}$

then L.H.S $= A \cup (B \cap C)$

$= \{1, 2, 4, 8\} \cup (\{2, 4, 6\} \cap \{3, 4, 5, 6\})$

$= \{1, 2, 4, 8\} \cup \{4, 6\} = \{1, 2, 4, 6, 8\}$

and R.H.S $= (A \cup B) \cap (A \cup C)$

$= (\{1, 2, 4, 8\} \cup \{2, 4, 6\}) \cap (\{1, 2, 4, 8\} \cup \{3, 4, 5, 6\})$

$= \{1, 2, 4, 6, 8\} \cap \{1, 2, 3, 4, 5, 6, 8\}$

$= \{1, 2, 4, 6, 8\}$

L.H.S = R.H.S

(f) Intersection is distributive over union of sets

To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Suppose $A = \{1, 2, 3, 4, 5, \cdots, 20\}$

$B = \{5, 10, 15, 20, 25, 30\}$

$C = \{3, 9, 15, 21, 27, 33\}$

L.H.S. $= A \cap (B \cup C)$

$= \{1, 2, 3, 4, 5, \cdots, 20\} \cap (\{5, 10, 15, 20, 25, 30\} \cup \{3, 9, 15, 21, 27, 33\})$

$= \{1,2,3,4,5, \cdots, 20\} \cap \{3, 5, 9, 10, 15, 20, 21, 25, 27, 30, 33\}$

$= \{3, 5, 9, 10, 15, 20\}$

R.H.S. $= (A \cap B) \cup (A \cap C)$

$= (\{1, 2, 3, 4, \cdots, 20\} \cap \{5, 10, 15, 20, 25, 30\})$

$\cup (\{1, 2, 3, 4, 5, \cdots, 20\} \cap \{3, 9, 15, 21, 27, 33\})$

$= \{5, 10, 15, 20\} \cup \{3, 9, 15\} = \{3, 5, 9, 10, 15, 20\}$

L.H.S. = R.H.S.

(g) De Morgan's Laws $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$

Suppose $U = \{1, 2, 3, 4, \cdots, 10\}$

$A = \{2, 4, 6, 8, 10\} \Rightarrow A' = \{1, 3, 5, 7, 9\}$

$B = \{1, 2, 3, 4, 5, 6\} \Rightarrow B' = \{7, 8, 9, 10\}$

Now consider $A \cap B = \{2, 4, 6, 8, 10\} \cap \{1, 2, 3, 4, 5, 6\}$

$= \{2, 4, 6\}$

Then L.H.S. $= (A \cap B)' = U - (A \cap B)$

$= \{1, 2, 3, 4, \cdots, 10\} - \{2, 4, 6\}$

$= \{1, 3, 5, 7, 8, 9, 10\}$

and R.H.S. $= A' \cup B'$

$= \{1, 3, 5, 7, 9\} \cup \{7, 8, 9, 10\}$

$= \{1, 3, 5, 7, 8, 9, 10\}$

L.H.S. = R.H.S.

$(A \cup B)' = A' \cap B'$

Suppose $U = \{1, 2, 3, 4, \cdots, 10\}$

$A = \{2, 4, 6, 8, 10\} \Rightarrow A' = \{1, 3, 5, 7, 9\}$

$B = \{1, 2, 3, 4, 5, 6\} \Rightarrow B' = \{7, 8, 9, 10\}$

Now consider $A \cup B = \{2, 4, 6, 8, 10\} \cup \{1, 2, 3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6, 8, 10\}$

L.H.S. $= (A \cup B)' = U - (A \cup B)$

$= \{1, 2, 3, 4, \cdots, 10\} - \{1, 2, 3, 4, 5, 6, 8, 10\}$

$= \{7, 9\}$

and R.H.S $= A' \cap B' = \{1, 3, 5, 7, 9\} \cap \{7, 8, 9, 10\}$

$= \{7, 9\}$

L.H.S. = R.H.S.

5.1.3 VENN DIAGRAM

British mathematician John Venn (1834 – 1923) introduced rectangle for a universal set U and its subsets A and B as closed figures inside this rectangle.

5.1.3(vi) Use Venn diagrams to represent:

(a) Union and intersection of sets

Disjoint sets | Overlapping sets

$A \quad B$

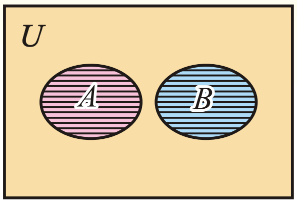


Fig. 1

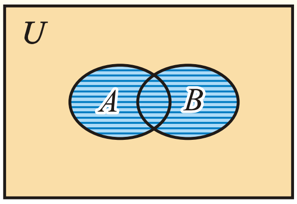


Fig. 2

$A \quad B$

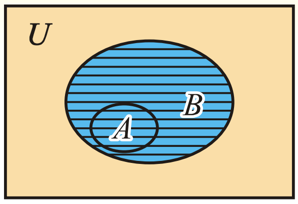


Fig. 3

$A \quad B$

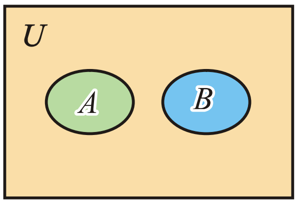


Fig. 4

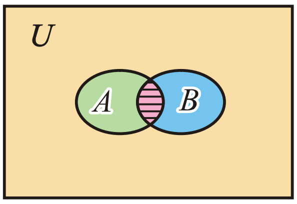


Fig. 5

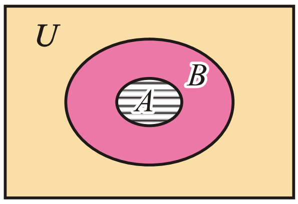
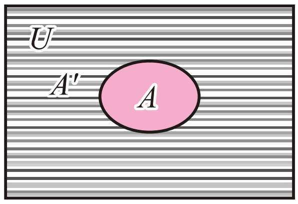


Fig. 6

(Regions shown by horizontal line segments in figures 1 to 6.)

(b) Complement of a set

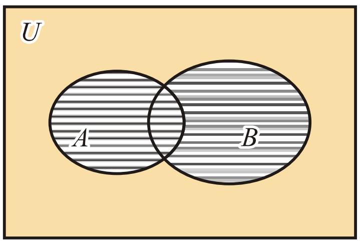
$U - A = A'$ is shown by horizontal line segments.



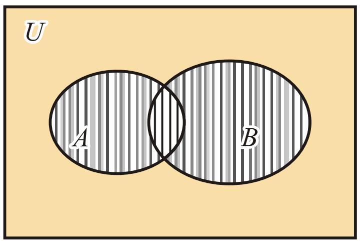
5.1.3 (vii) Use Venn diagram to verify:

(a) Commutative law for union and intersection of sets

$A \quad B$



$B \quad A$

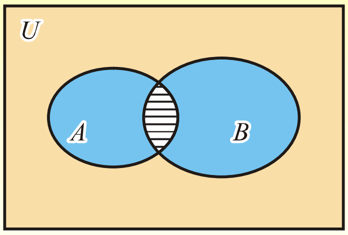


$A \cup B$ is shown by horizontal line segments.

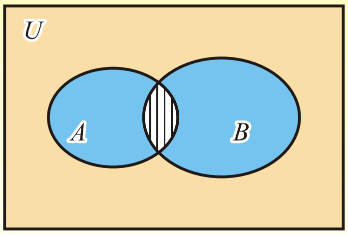
$B \cup A$ is shown by vertical line segments.

The regions shown in both cases are equal. Thus $A \cup B = B \cup A$.

A B



B A



$A \cap B$ is shown by horizontal line segments. $B \cap A$ is shown by vertical line segments.

The regions shown in both cases are equal. Thus $A \cap B = B \cap A$.

(b) De Morgan's laws

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(i) $(A \cup B)' = A' \cap B'$

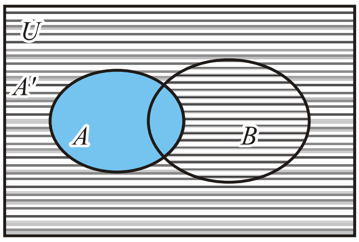


Fig. 1: A' is shown by horizontal line segments

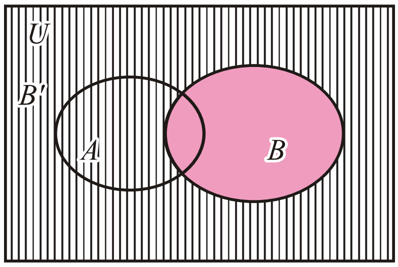


Fig. 2: B' is shown by vertical line segments

A' B'

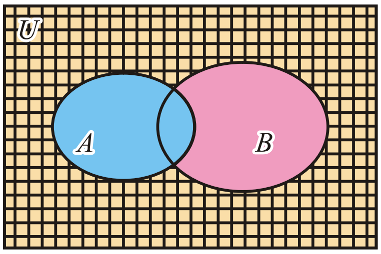


Fig. 3: $A' \cap B'$ is shown by squares

$(A$ $B)'$

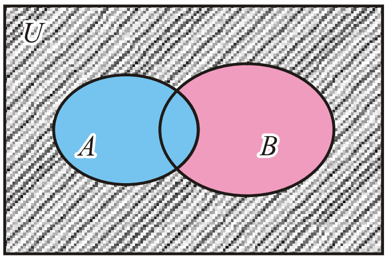


Fig. 4: $(A \cup B)'$ is shown by slanting line segments

Regions shown in Fig. 3 and Fig. 4 are equal.
Thus $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

A' B'

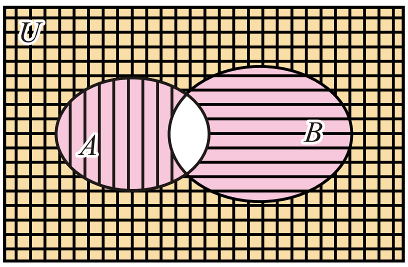


Fig. 5: $A' \cup B'$ is shown by squares, horizontal and vertical line segments.

$(A \cap B)'$

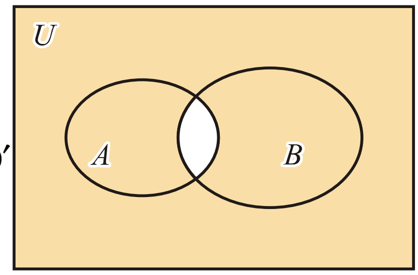


Fig. 6: $U - (A \cap B) = (A \cap B)'$ is shown by shading.

Regions shown in Fig. 5 and Fig. 6 are equal.

Thus $(A \cap B)' = A' \cup B'$

(c) Associative law:

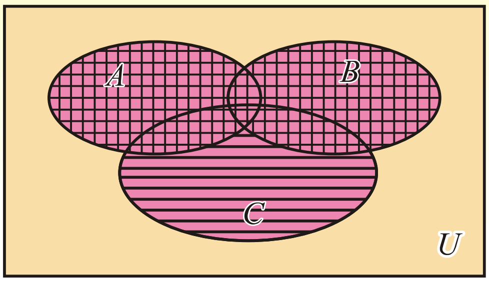


Fig. 1

$(A \cup B) \cup C$ is shown in the above figure.

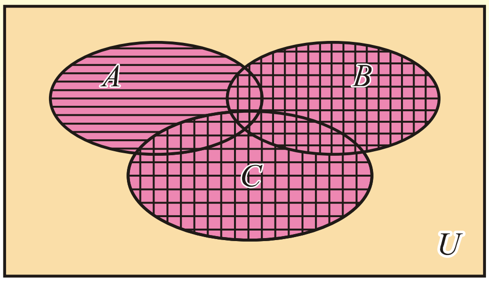


Fig. 2

$A \cup (B \cup C)$ is shown in the above figure.

Regions shown in fig. 1 and fig. 2 by different ways are equal.

Thus $(A \cup B) \cup C = A \cup (B \cup C)$

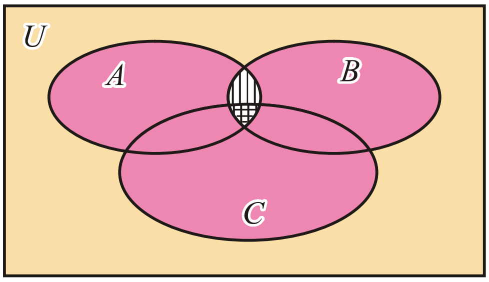


Fig. 3

$(A \cap B) \cap C$ is shown in figure 3 by double crossing line segments

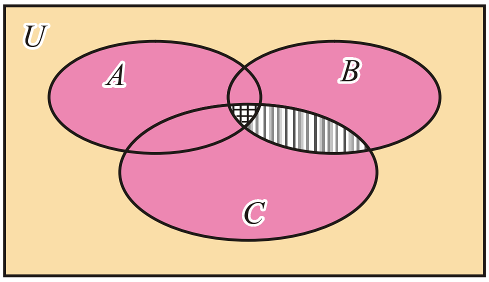


Fig. 4

$A \cap (B \cap C)$ is shown in figure 4 by double crossing line segments

Regions shown in Fig. 3 and fig. 4 are equal.

Thus $(A \cap B) \cap C = A \cap (B \cap C)$

(d) Distributive law:

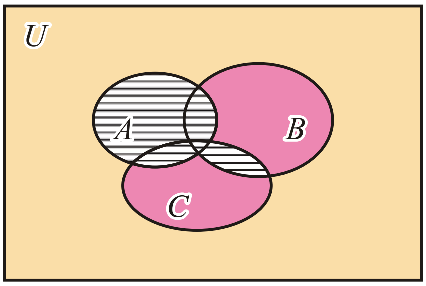


Fig. 1: $A \cup (B \cap C)$ is shown by horizontal line segments in the above figure.

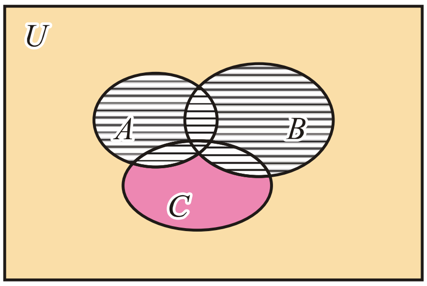


Fig. 2: $A \cup B$ is shown by horizontal line segments in the above figure.

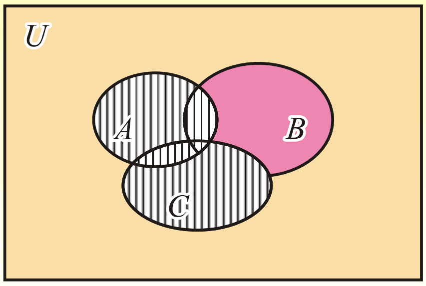


Fig. 3: $A \cup C$ is shown by vertical line segments in Fig. 3.

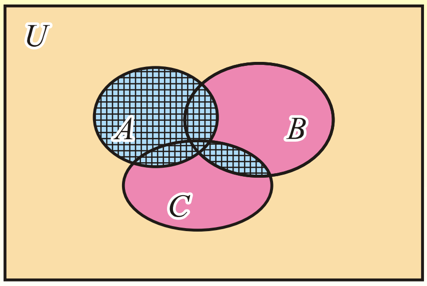


Fig. 4: $(A \cup B) \cap (A \cup C)$ is shown by double crossing line segments in Fig. 4.

Regions shown in Fig. 1 and Fig. 4 are equal.

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

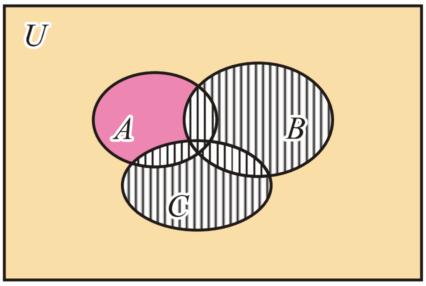


Fig. 5: $B \cup C$ is shown by vertical line segments in Fig. 5.

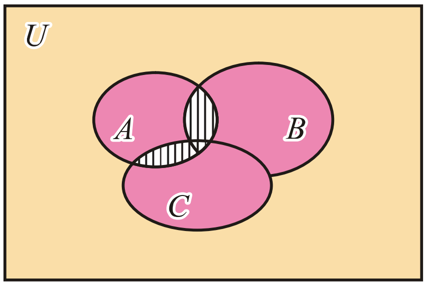


Fig. 6: $A \cap (B \cup C)$ is shown in Fig. 6 by vertical line segments.

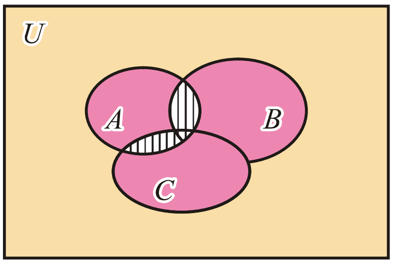


Fig. 7: $(A \cap B) \cup (A \cap C)$ is shown in Fig. 7 by slanting line segments.

Regions displayed in Fig. 6 and Fig. 7 are equal.

Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

EXERCISE 5.3

1. If $U = \{1, 2, 3, 4, \cdots, 10\}$
$A = \{1, 3, 5, 7, 9\}$
$B = \{1, 4, 7, 10\}$,
then verify the following questions.
(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$
(iii) $(A \cup B)' = A' \cap B'$ (iv) $(A \cap B)' = A' \cup B'$
(v) $(A - B)' = A' \cup B$ (vi) $(B - A)' = B' \cup A$

2. If $U = \{1, 2, 3, 4, \cdots, 10\}$
$A = \{1, 3, 5, 7, 9\}$; $B = \{1, 4, 7, 10\}$; $C = \{1, 5, 8, 10\}$
then verify the following:
(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

3. If $U = N$; then verify De-Morgan's laws by using $A = \phi$ and $B = P$.

4. If $U = \{1, 2, 3, 4, \cdots, 10\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 4, 5, 8\}$, then prove the following questions by Venn diagram:
(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$
(iii) $(A \cup B)' = A' \cap B'$ (iv) $(A \cap B)' = A' \cup B'$
(v) $(A - B)' = A' \cup B$ (vi) $(B - A)' = B' \cup A$

5.1.4 (viii) Ordered pairs and Cartesian product:

5.1.4(a) Ordered pairs:

Any two numbers x and y, written in the form (x, y) is called an ordered pair. In an ordered pair (x, y), the order of numbers is important, that is, x is the first co-ordinate and y is the second co-ordinate. For example, (3, 2) is different from (2, 3).

It is obvious that $(x, y) \neq (y, x)$ unless $x = y$.

Note that $(x, y) = (s, t)$, iff $x = s$ and $y = t$

5.1.4(b) Cartesian product:

Cartesian product of two non-empty sets A and B denoted by $A \times B$ consists of all ordered pairs (x, y) such that $x \in A$ and $y \in B$.

Example: If $A = \{1, 2, 3\}$ and $B = \{2, 5\}$, then find $A \times B$ and $B \times A$.

Solution: $A \times B = \{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$

Since set A has 3 elements and set B has 2 elements.

Hence product set $A \times B$ has $3 \times 2 = 6$ ordered pairs.

We note that $B \times A = \{(2, 1), (2, 2), (2, 3), (5, 1), (5, 2), (5, 3)\}$

Evidently $A \times B \neq B \times A$.

EXERCISE 5.4

1. If $A = \{a, b\}$ and $B = \{c, d\}$, then find $A \times B$ and $B \times A$.
2. If $A = \{0, 2, 4\}$, $B = \{-1, 3\}$, then find $A \times B$, $B \times A$, $A \times A$, $B \times B$
3. Find a and b, if
   (i) $(a - 4, b - 2) = (2, 1)$ (ii) $(2a + 5, 3) = (7, b - 4)$
   (iii) $(3 - 2a, b - 1) = (a - 7, 2b + 5)$
4. Find the sets X and Y, if $X \times Y = \{(a, a), (b, a), (c, a), (d, a)\}$
5. If $X = \{a, b, c\}$ and $Y = \{d, e\}$, then find the number of elements in
   (i) $X \times Y$ (ii) $Y \times X$ (iii) $X \times X$

5.2 Binary relation

If A and B are any two non-empty sets, then a subset $R \subseteq A \times B$ is called **binary relation** from set A into set B, because there exists some relationship between first and second element of each ordered pair in R.

Domain of relation denoted by *Dom R* is the set consisting of all the first elements of each ordered pair in the relation.

Range of relation denoted by *Rang R* is the set consisting of all the second elements of each ordered pair in the relation.

Example 1: Suppose $A = \{2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$

Form a relation $R : A \to B = \{x\,R\,y$ such that $y = 2x$ for $x \in A, y \in B\}$

$\Rightarrow$ $R = \{(2, 4), (3, 6), (4, 8)\}$

Dom $R = \{2, 3, 4\} \subseteq A$ and Rang $R = \{4, 6, 8\} \subseteq B$.

Example 2: Suppose $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 5\}$

Form a relation $R : A \to B = \{x\,R\,y$ such that $x + y = 6$ for $x \in A, y \in B\}$

$\Rightarrow$ $R = \{(1, 5), (3, 3), (4, 2)\}$

Dom $R = \{1, 3, 4\} \subseteq A$ and Rang $R = \{2, 3, 5\} \subseteq B$

5.3 Function or Mapping:

5.3. (i) Suppose A and B are two non-empty sets, then relation $f : A \to B$ is called a function if (i) Dom $f = A$ (ii) every $x \in A$ appears in one and only one ordered pair in f.

Alternate Definition:

Suppose A and B are two non-empty sets, then relation $f : A \to B$ is called a function if (i) Dom $f = A$ (ii) $\forall\, x \in A$ we can associate some unique image element $y = f(x) \in B$.

Domain, Co-domain and Range of Function:

If $f : A \to B$ is a function, then A is called the domain of f and B is called the co-domain of f.

Domain f is the set consisting of all first elements of each ordered pair in f and range f is the set consisting of all second elements of each ordered pair in f.

Example: Suppose $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

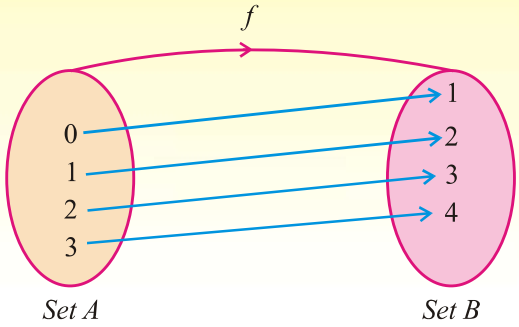
Define a function $f : A \to B$

$f = \{(x, y) \mid y = x + 1 \;\forall\; x \in A,\, y \in B\}$

$f = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$

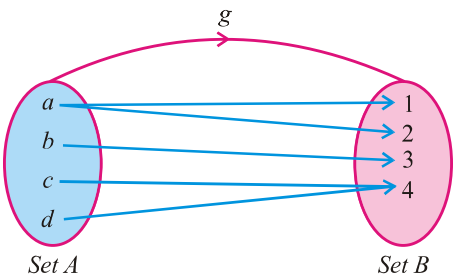
Dom $f = \{0, 1, 2, 3\} = A$

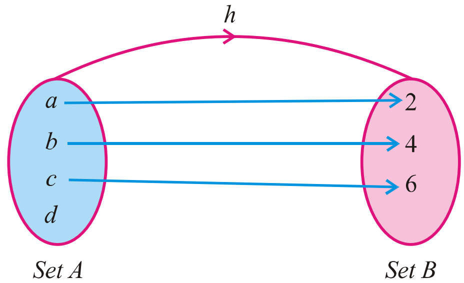
Rang $f = \{1, 2, 3, 4\} \subseteq B$.



The following are the examples of relations but not functions.

g is not a function, because an element $a \in A$ has two images in set B and h is not a function because an element $d \in A$ has no image in set B.





5.3(ii) Demonstrate the following:

(a) Into function:

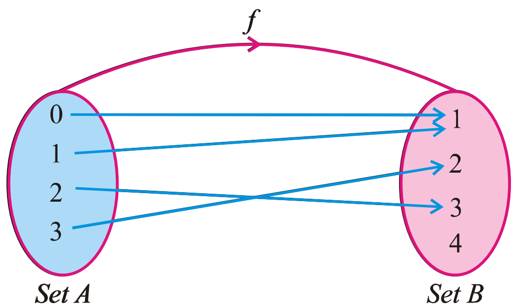
A function $f : A \to B$ is called an into function, if at least one element in B is not an image of some element of set A *i.e.*, Range of $f \subset$ set B.

For example, we define a function $f : A \to B$ such that

$f = \{(0, 1), (1, 1), (2, 3), (3, 2)\}$

where $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$

f is an into function.



(b) One-one function:

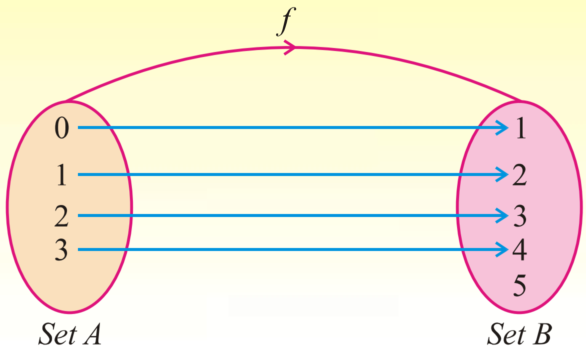
A function $f : A \to B$ is called one-one function, if all distinct elements of A have distinct images in B, *i.e.*, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \in A$ or $\forall\, x_1 \neq x_2 \in A \Rightarrow f(x_1) \neq f(x_2)$

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, then we define a function $f : A \rightarrow B$ such that

$f = \{(x, y) \mid y = x + 1, \forall\, x \in A\,, y \in B\}$.

$= \{(0, 1,), (1, 2), (2, 3), (3, 4)\}$

f is one-one function.



(c) Into and one-one function: (injective function)

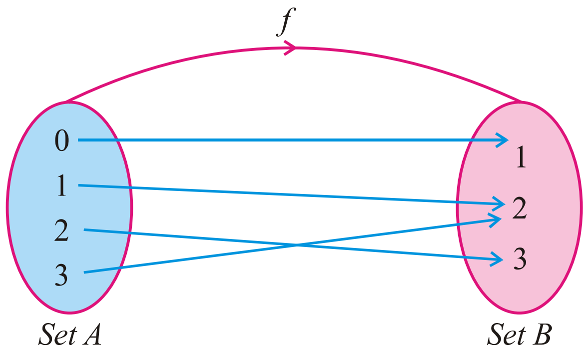
The function f discussed in (b) is also an into function. Thus f is an into and one-one function.

(d) An onto or surjective function:

A function $f : A \rightarrow B$ is called an onto function, if every element of set B is an image of at least one element of set A *i.e.*, Range of $f = B$.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $f : A \rightarrow B$ such that $f = \{(0, 1), (1, 2), (2, 3), (3, 2)\}$. Here Rang $f = \{1, 2, 3\} = B$.

Thus f so defined is an onto function.



(e) Bijective function or one to one correspondence:

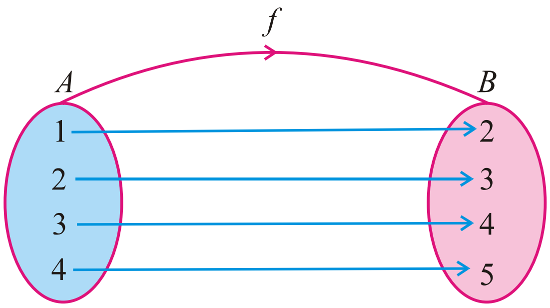
A function $f : A \rightarrow B$ is called bijective function iff function f is one-one and onto.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$

We define a function $f : A \rightarrow B$ such that

$f = \{(x, y) \mid y = x + 1, \forall\, x \in A, y \in B\}$

Then $f = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$



Evidently this function is one-one because distinct elements of A have distinct images in B. This is an onto function also because every element of B is the image of atleast one element of A.

Note: (1) Every function is a relation but converse may not be true.
(2) Every function may not be one-one.
(3) Every function may not be onto.

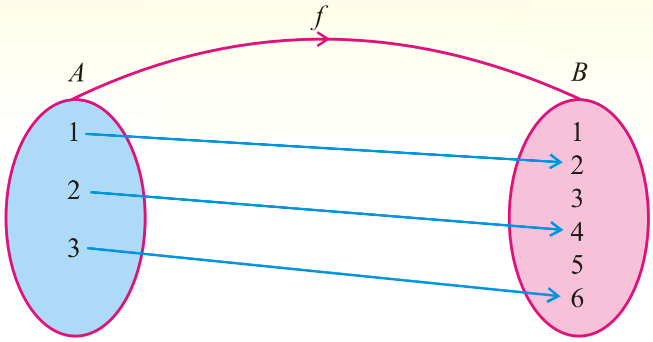
Example: Suppose $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5, 6\}$

We define a function $f: A \rightarrow B = \{(x, y) \mid y = 2x, \forall\, x \in A, y \in B\}$

Then $f = \{(1, 2), (2, 4), (3, 6)\}$

Evidently this function is one-one but not an onto



5.3(iii) Examine whether a given relation is a function:

A relation in which each $x \in$ its domain, has a unique image in its range, is a function.

5.3(iv) Differentiate between one-to-one correspondence and one-one function:

A function f from set A to set B is one-one if distinct elements of A has distinct images in B. The domain of f is A and its range is contained in B.

In one-to-one correspondence between two sets A and B, each element of either set is assigned with exactly one element of the other set. If the sets A and B are finite, then these sets have the same number of elements, that is, $n(A) = n(B)$.

EXERCISE 5.5

1. If $L = \{a, b, c\}$, $M = \{3, 4\}$, then find two binary relations of $L \times M$ and $M \times L$.
2. If $Y = \{-2, 1, 2\}$, then make two binary relations for $Y \times Y$. Also find their domain and range.
3. If $L = \{a, b, c\}$ and $M = \{d, e, f, g\}$, then find two binary relations in each:

   (i) $L \times L$ (ii) $L \times M$ (iii) $M \times M$
4. If set M has 5 elements, then find the number of binary relations in M.
5. If $L = \{x \mid x \in N \wedge x \leq 5\}$, $M = \{y \mid y \in P \wedge y < 10\}$, then make the following relations from L to M

   (i) $R_1 = \{(x, y) \mid y < x\}$ (ii) $R_2 = \{(x, y) \mid y = x\}$

   (iii) $R_3 = \{(x, y) \mid x + y = 6\}$ (iv) $R_4 = \{(x, y) \mid y - x = 2\}$

   Also write the domain and range of each relation.
6. Indicate relations, into function, one-one function, onto function, and bijective function from the following. Also find their domain and the range.

(i) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

(ii) $R_2 = \{(1, 2), (2, 1), (3, 4), (3, 5)\}$

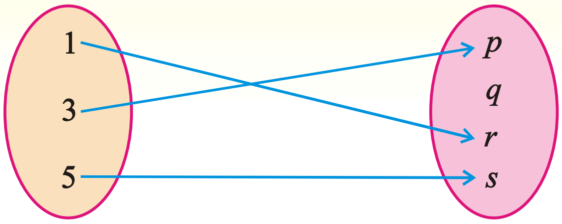
(iii) $R_3 = \{(b, a), (c, a), (d, a)\}$

(iv) $R_4 = \{(1, 1), (2, 3), (3, 4), (4, 3), (5, 4)\}$

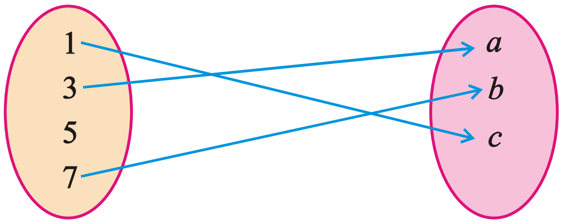
(v) $R_5 = \{(a, b), (b, a), (c, d), (d, e)\}$

(vi) $R_6 = \{(1, 2), (2, 3), (1, 3), (3, 4)\}$

(vii) $R_7 =$



(viii) $R_8 =$



MISCELLANEOUS EXERCISE - 5

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick mark (✓) the correct answer.

(i) A collection of well-defined objects is called

(a) subset	(b) power set
(c) set	(d) none of these

(ii) A set $Q = \left\{ \frac{a}{b} \mid a, b \in Z \wedge b \neq 0 \right\}$ is called a set of

(a) Whole numbers	(b) Natural numbers
(c) Irrational numbers	(d) Rational numbers

(iii) The different number of ways to describe a set are

(a) 1	(b) 2
(c) 3	(d) 4

(iv) A set with no element is called

(a) Subset	(b) Empty set
(c) Singleton set	(d) Super set

(v) The set $\{ x \mid x \in W \wedge x \leq 101 \}$ is

(a) Infinite set	(b) Subset
(c) Null set	(d) Finite set

(vi) The set having only one element is called

(a) Null set (b) Power set
(c) Singleton set (d) Subset

(vii) Power set of an empty set is

(a) ϕ (b) $\{a\}$
(c) $\{\phi, \{a\}\}$ (d) $\{\phi\}$

(viii) The number of elements in power set $\{1, 2, 3\}$ is

(a) 4 (b) 6
(c) 8 (d) 9

(ix) If $A \subseteq B$, then $A \cup B$ is equal to

(a) A (b) B
(c) ϕ (d) none of these

(x) If $A \subseteq B$, then $A \cap B$ is equal to

(a) A (b) B
(c) ϕ (d) none of these

(xi) If $A \subseteq B$, then $A - B$ is equal to

(a) A (b) B
(c) ϕ (d) $B - A$

(xii) $(A \cup B) \cup C$ is equal to

(a) $A \cap (B \cup C)$ (b) $(A \cup B) \cap C$
(c) $A \cup (B \cup C)$ (d) $A \cap (B \cap C)$

(xiii) $A \cup (B \cap C)$ is equal to

(a) $(A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cap C)$
(c) $(A \cap B) \cup (A \cap C)$ (d) $A \cup (B \cup C)$

(xiv) If A and B are disjoint sets, then $A \cup B$ is equal to

(a) A (b) B
(c) ϕ (d) $B \cup A$

(xv) If number of elements in set A is 3 and in set B is 4, then number of elements in $A \times B$ is

(a) 3 (b) 4
(c) 12 (d) 7

(xvi) If number of elements in set A is 3 and in set B is 2, then number of binary relations in $A \times B$ is

(a) 2^3 (b) 2^6
(c) 2^8 (d) 2^2

(xvii) The domain of $R = \{(0, 2), (2, 3), (3, 3), (3, 4)\}$ is

(a) $\{0, 3, 4\}$ (b) $\{0, 2, 3\}$
(c) $\{0, 2, 4\}$ (d) $\{2, 3, 4\}$

(xviii) The range of $R = \{(1, 3), (2, 2), (3, 1), (4, 4)\}$ is

(a) $\{1, 2, 4\}$ (b) $\{3, 2, 4\}$
(c) $\{1, 2, 3, 4\}$ (d) $\{1, 3, 4\}$

(xix) Point (–1, 4) lies in the quadrant

(a) I (b) II

(c) III (d) IV

(xx) The relation {(1, 2), (2, 3), (3, 3), (3, 4)} is

(a) onto function (b) into function

(c) not a function (d) one-one function

2. Write short answers of the following questions.

(i) Define a subset and give one example.

(ii) Write all the subsets of the set $\{a, b\}$

(iii) Show $A \cap B$ by Venn diagram. When $A \subseteq B$

(iv) Show by Venn diagram $A \cap (B \cup C)$.

(v) Define intersection of two sets.

(vi) Define a function.

(vii) Define one-one function.

(viii) Define an onto function.

(ix) Define a bijective function.

(x) Write De Morgan's laws.

3. Fill in the blanks

(i) If $A \subseteq B$, then $A \cup B =$ _________.

(ii) If $A \cap B = \phi$ then A and B are _______.

(iii) If $A \subseteq B$ and $B \subseteq A$ then ___________.

(iv) $A \cap (B \cup C) =$ ___________.

(v) $A \cup (B \cap C) =$ _________.

(vi) The complement of U is ____________.

(vii) The complement of ϕ is ____________.

(viii) $A \cap A^c =$ ____________.

(ix) $A \cup A^c =$ ____________.

(x) The set $\{x \mid x \in A$ and $x \notin B\} =$ ____________.

(xi) The point (–5, –7) lies in ____________ quadrant.

(xii) The point (4, –6) lies in ____________ quadrant.

(xiii) The ***y*** co-ordinate of every point is ____________ on-x-axis.

(xiv) The ***x*** co-ordinate of every point is ____________ on-y-axis.

(xv) The domain of $\{(a, b), (b, c), (c, d)\}$ is ____________.

(xvi) The range of $\{(a, a), (b, b), (c, c)\}$ is ____________.

(xvii) Venn-diagram was first used by ____________.

(xviii) A subset of $A \times A$ is called the ____________ in A.

(xix) If $f: A \longrightarrow B$ and range of $f = B$, then f is an ____________ function.

(xx) The relation $\{(a, b), (b, c), (a, d)\}$ is ____________ a function.

SUMMARY

- A set is the **well defined collection** of objects with some common properties.
- Union of two sets A and B denoted by $A \cup B$ is the set **containing elements** which either belong to A or to B or to both.
- Intersection of two sets A and B denoted by $A \cap B$ is the set of **common elements** of both A and B. In symbols $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
- The set difference of B and A denoted by $B - A$ is the set of all those elements of B which **do not belong to A.**
- Complement of a set A w.r.t. **universal** set U denoted by $A^C = A' = U - A$ contains all those elements of U which **do not belong to A.**
- British mathematician John Venn (1834 – 1923) introduced rectangle for a universal set U and its subsets A and B as **closed figures** inside this rectangle.
- An ordered pair of elements is written according to a **specific order** for which the order of elements is strictly maintained.
- Cartesian product of two non-empty sets A and B denoted by $A \times B$ consists of all **ordered pairs** (x, y) such that $x \in A, y \in B$.
- If A and B are any two non-empty sets, then a non empty subset $R \subseteq A \times B$ is called binary relation from set A into set B.
- If A and B are two non empty sets, then **relation** $f : A \to B$ is called a function if (i) Dom f = set A (ii) every $x \in A$ appears in one and only one ordered pair $\in f$.
- Dom f is the set consisting of all **first elements** of each ordered pair $\in f$ and Rang of f is the set consisting of all **second elements** of each ordered pair $\in f$.
- A function $f : A \to B$ is called an into function if at least one element in B is not an image of some element of set A *i.e.*, **range of $f \subseteq B$.**
- A function $f : A \to B$ is called an onto function if every element of set B is an image of at least one element of set A *i.e.*, **range of $f = B$**.
- A function $f : A \to B$ is called one-one function if all **distinct elements** of A have distinct images in B
- A function $f : A \to B$ is called bijective function iff function f is **one-one and onto**.