Unit-7

INTRODUCTION TO TRIGONOMETRY

In this unit, students will learn how to

- *measure an angle in degree, minute and second.*
- *convert an angle given in degrees, minutes and seconds into decimal form and vice versa.*
- *define a radian (measure of an angle in circular system) and prove the relationship between radians and degrees.*
- *establish the rule $l = r\theta$, where r is the radius of the circle, l the length of circular arc and θ the central angle measured in radians.*
- *prove that the area of a sector of a circle is $\frac{1}{2}r^2\theta$.*
- *define and identify:*
  - *general angle (coterminal angles),*
  - *angle in standard position.*
- *recognize quadrants and quadrantal angles.*
- *define trigonometric ratios and their reciprocals with the help of a unit circle.*
- *recall the values of trigonometric ratios for 45°, 30°, 60°.*
- *recognise signs of trigonometric ratios in different quadrants.*
- *find values of remaining trigonometric ratios if one trigonometric ratio is given.*
- *calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°.*
- *prove the trigonometric identities and apply them to show different trigonometric relations.*
- *find angle of elevation and depression.*
- *solve real life problems involving angle of elevation and depression.*

7.1 Measurement of an Angle

An **angle** is defined as the union of two non-collinear rays with some common end point. The rays are called **arms** of the angle and the common end point is known as **vertex** of the angle.

It is easy if we make an angle by rotating a ray from one position to another. When we form an angle in this way, the original position of the ray is called **initial side** and final position of the ray is called the **terminal side** of the angle. If the rotation of the ray is anti-clockwise or clockwise, the angle has positive or negative measure respectively.

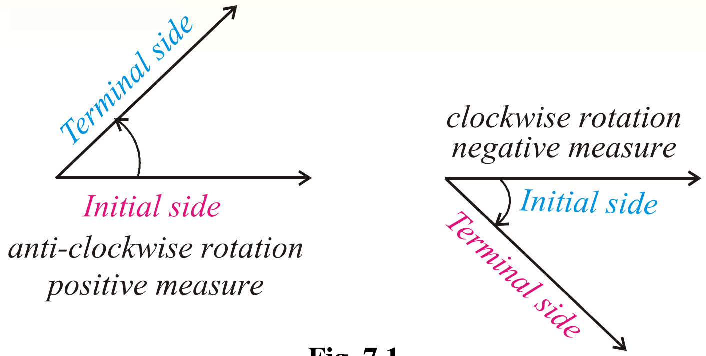


Fig. 7.1

7.1(i) Measurement of an angle in sexagesimal system (degree, minute and second)

Degree: We divide the circumference of a circle into 360 equal arcs. The angle subtended at the centre of the circle by one arc is called one degree and is denoted by 1^{o}.

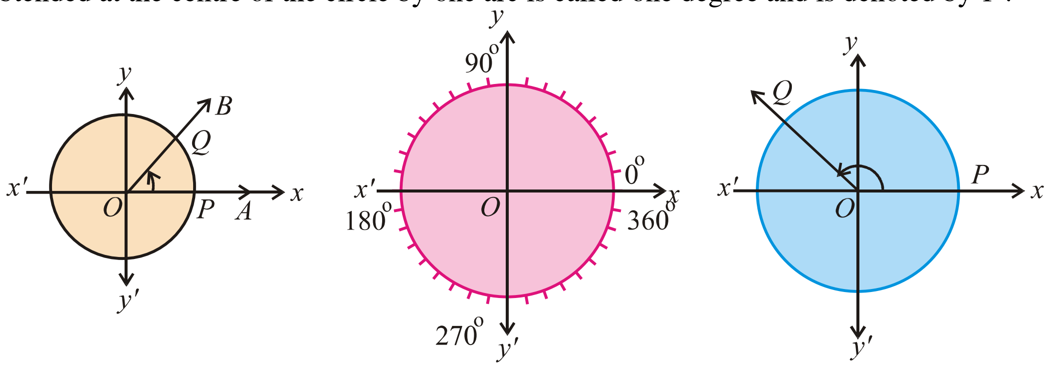


Fig. 7.1.1

The symbols 1^{o}, $1'$ and $1''$ are used to denote a degree, a minute and a second respectively.

Thus 60 seconds $(60'')$ make one minute $(1')$

60 minutes $(60')$ make one degree (1^{o})

90 degrees (90^{o}) make one right angle

360 degrees (360^{o}) make 4 right angles

An angle of 360^{o} denotes a complete circle or one revolution. We use coordinate system to locate any angle to a **standard position,** where its initial side is the positive x-axis and its vertex is the origin.

Example: Locate (*a*) -45° (*b*) 120° (*c*) 45° (*d*) -270°

Solution: The angles are shown in Figure 7 1 2

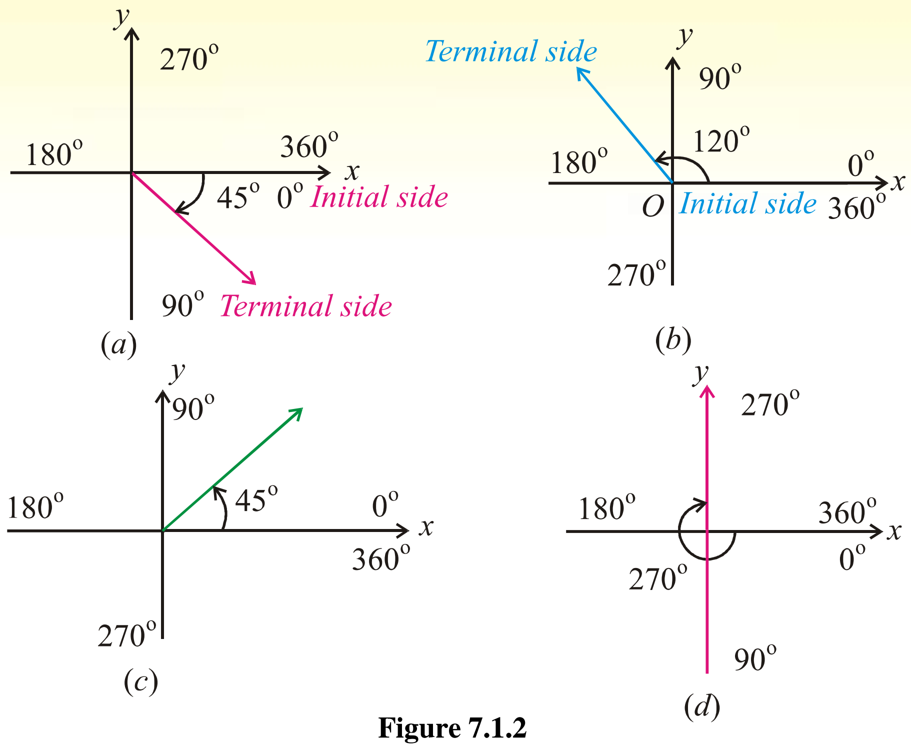


Figure 7.1.2

7.1(ii) Conversion of an angle given in $D^o\ M'S''$ form into decimal form and vice versa

The conversion is explained through examples.

Example 1: (i) Convert $25^\circ 30'$ to decimal degrees.

(ii) Convert 32.25° to $D^o\ M'S''$ form.

Solution:

(i) $25^\circ 30' = 25^\circ + \left(\frac{30}{60}\right)^\circ = 25^\circ + 0.5^\circ = 25.5^\circ$

(ii) $32.25^\circ = 32^\circ + 0.25^\circ = 32^\circ + \left(\frac{25}{100}\right)^\circ$

$= 32^\circ + \frac{1^\circ}{4} = 32^\circ + \left(\frac{1}{4}\times 60\right)' = 32^\circ\ 15'$

Example 2: Convert $12^\circ 23' 35''$ to decimal degrees correct to three decimal places.

Solution: $12^\circ 23' 35'' = 12^\circ + \frac{23^\circ}{60} + \frac{35^\circ}{60\times 60} = \left(12^\circ + \frac{23^\circ}{60} + \frac{35^\circ}{3600}\right)$

$\approx\ 12^\circ + 0.3833^\circ + 0.00972^\circ$

$\approx 12.3930^{\circ} = 12.393^{\circ}$

Example 3: Convert 45.36° to $D^{o}\, M'S''$ form.

Solution: $(45.36)^{\circ} = 45^{\circ} + (0.36)^{\circ}$

$$= 45^{\circ} + \left(\frac{36}{100}\right)^{\circ} = 45^{\circ} + \left(\frac{9}{25} \times 60'\right)$$

$$= 45^{\circ} + 21.6' = 45^{\circ} + 21' + (0.6 \times 60)''$$

$$= 45^{\circ}21'36''$$

7.1(iii) Radian measure of an angle (circular system).

Another system of measurement of an angle known as circular system is of most importance and is used in all the higher branches of Mathematics.

Radian: The angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle is called one **Radian**.

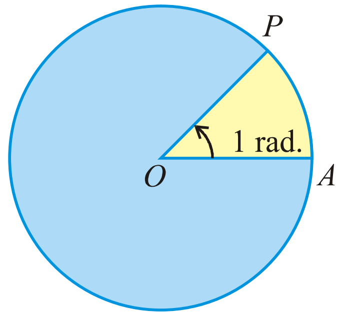


Fig. 7.1.3

Consider a circle of radius r whose centre is O. From any point A on the circle cut off an arc AP whose length is equal to the radius of the circle. Join O with A and O with P. The $\angle AOP$ is one radian. This means that when length of arc $\overset{\frown}{AP}$ = length of radius $\overline{OA}$ then $m\angle AOP = 1$ radian

Relationship between radians and degrees

We know that circumference of a circle is $2\pi r$ where r is the radius of the circle. Since a circle is an arc whose length is $2\pi r$. The radian measure of an angle that form a complete circle is $\dfrac{2\pi r}{r} = 2\pi$

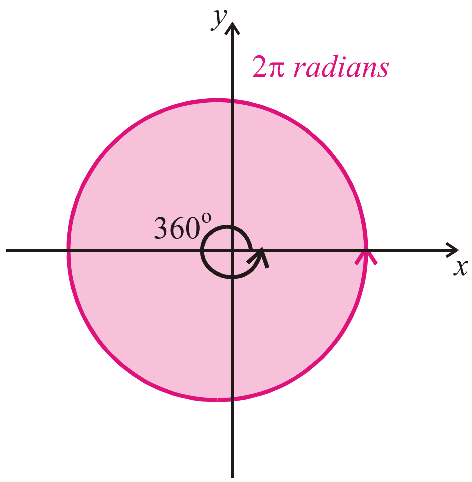


Fig. 7.1.4

From this, we see that $360^{\circ} = 2\pi$ radians

or $180^{\circ} = \pi$ radians …… (i)

Using this relation, we can convert degrees into radians and radians into degrees as follows:

$$180^{\circ} = \pi \text{ radians} \Rightarrow 1^{\circ} = \frac{\pi}{180} \text{ radian},$$

$$x^{o} = x \,.\, 1^{\circ} = x\left(\frac{\pi}{180}\right) \text{radian} \quad \text{...... (ii)}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ},\ y \text{ radian} = y\left(\frac{180}{\pi}\right) \text{degrees} \quad \text{...... (iii)}$$

Some special angles in degree and radians.

$180^\circ = 1\,(180^\circ) = \pi$ radians

$90^\circ = \frac{1}{2}(180^\circ) = \frac{\pi}{2}$ radians

$60^\circ = \frac{1}{3}(180^\circ) = \frac{\pi}{3}$ radians

$45^\circ = \frac{1}{4}(180^\circ) = \frac{\pi}{4}$ radians

$30^\circ = \frac{1}{6}(180^\circ) = \frac{\pi}{6}$ radians

$270^\circ = \frac{3}{2}(180^\circ) = \frac{3\pi}{2}$ radians

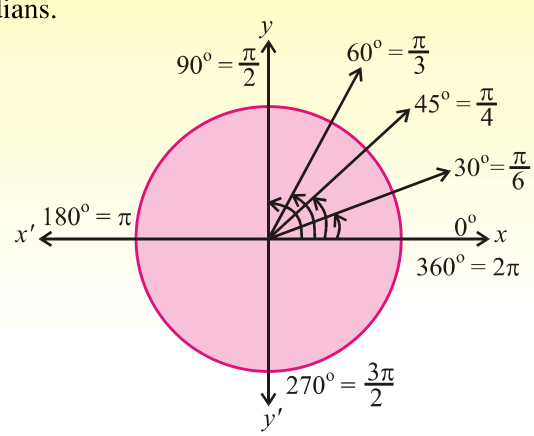


Fig. 7.1.5

Example 4: Convert the following angles into radian measure:

(*a*) 15° (*b*) $124^\circ 22'$

Solution: (*a*) $15^\circ = 15\left(\frac{\pi}{180}\text{ rads}\right)$ by using (*i*)

$= \frac{\pi}{12}$ radians

(*b*) $124^\circ 22' = \left(124 + \frac{22}{60}\right)^\circ = (124.3666)\left(\frac{\pi}{180}\right)$ radians

≈ 2.171 radians

Example 5: Express the following into degree.

(*a*) $\frac{2\pi}{3}$ radians (*b*) 6.1 radians

Solution:

(*a*) $\frac{2\pi}{3}$ radians $= \frac{2\pi}{3}\left(\frac{180}{\pi}\right)$ degrees

$= 120^\circ$

(*b*) 6.1 radians $= (6.1)\left(\frac{180}{\pi}\right)$ degrees

$= 6.1\,(57.295779) = 349.5043$ degrees

Remember that:

1 radian $\approx \left(\frac{180}{3.1416}\right)^\circ \approx 57.295795^\circ \approx 57^\circ 17' 45''$, $\quad 1^\circ \approx \frac{3.1416}{180} \approx 0.0175$ radians

EXERCISE 7.1

1. Locate the following angles:

   (i) 30° (ii) $22\frac{1}{2}^\circ$ (iii) 135° (iv) 225°

   (v) -60° (vi) -120° (vii) -150° (viii) -225°

2. Express the following sexagesimal measures of angles in decimal form.

   (i) $45^\circ 30'$ (ii) $60^\circ 30' 30''$ (iii) $125^\circ 22' 50''$

3. Express the following into $D^\circ\, M'\, S''$ form.

   (i) 47.36° (ii) 125.45° (iii) 225.75° (iv) -22.5°

   (v) -67.58° (vi) 315.18°

4. Express the following angles into radians.

   (i) 30° (ii) $(60)^\circ$ (iii) 135° (iv) 225° (v) -150°

   (vi) -225° (vii) 300° (viii) 315°

5. Convert each of following to degrees.

   (i) $\frac{3\pi}{4}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{7\pi}{8}$ (iv) $\frac{13\pi}{16}$ (v) 3

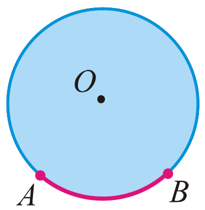
   (vi) 4.5 (vii) $\frac{-7\pi}{8}$ (viii) $-\frac{13}{16}\pi$

7.2 Sector of a Circle

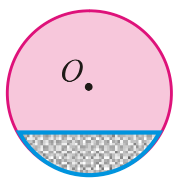
(i) A part of the circumference of a circle is called an **arc**.

(ii) A part of the circle bounded by an arc and a chord is called **segment of a circle**.

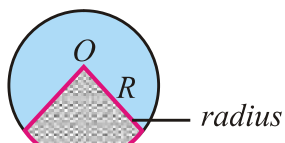
(iii) A part of the circle bounded by the two radii and an arc is called **sector of the circle**.



an arc



a segment



a sector

7.2(i) To establish the rule $l = r\,\theta$, where r is the radius of the circle, l the length of circular arc and θ the central angle measured in radians.

Let an arc AB denoted by l subtends an angle θ radian at the centre of the circle. It is a fact of plane geometry that measure of central angles of the arcs of a circle are proportional to the lengths of their arcs.

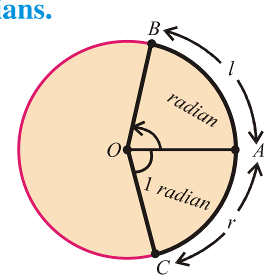
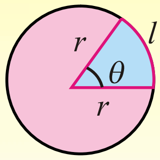


Fig. 7.2.1

$$\frac{m\angle AOB}{m\angle AOC} = \frac{m\widehat{AB}}{m\widehat{AC}}$$



$$\Rightarrow \quad \frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{l}{r} \quad \Rightarrow \quad \frac{l}{r} = \theta \quad \text{or} \quad l = r\theta$$

Example 1: In a circle of radius 10m,

(a) find the length of an arc intercepted by a central angle of 1.6 radian.

(b) find the length of an arc intercepted by a central angle of 60^o.

Solution: (a) Here $\theta = 1.6$ radian , $r = 10$m and $l = ?$

$\because$ Since $l = r\,\theta \Rightarrow l = 10 \times 1.6 = 16$ m

(b) $\theta = 60^o = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ rad.

$\because$ $l = r\,\theta = 10 \times \frac{\pi}{3} = \frac{10\pi}{3}$ m.

Example 2: Find the distance travelled by a cyclist moving on a circle of radius 15m, if he makes 3.5 revolutions.

Solution: 1 revolution = 2π radians

3.5 revolution = $2\pi \times 3.5$ m

Distance traveled = $l = r\,\theta$

$$l = 15 \times 2\pi \times 3.5 = 105\pi \text{ m}$$

7.2(ii) Area of a circular sector

Consider a circle of radius r units and an arc of length l units, subtends an angle θ at O.

Area of the circle = πr^2

Angle of the circle = 2π

Angle of the sector = θ radian

Then by elementary geometry we can use the proportion,

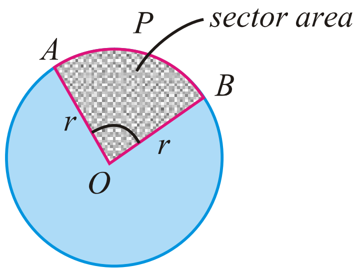


Fig. 7.2.2

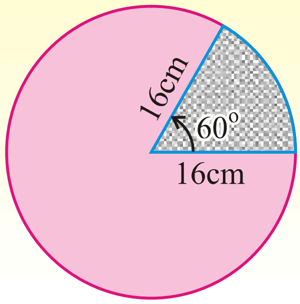
$$\frac{\text{area of sector } AOBP}{\text{area of circle}} = \frac{\text{angle of the sector}}{\text{angle of the circle}}$$

or $$\frac{\text{area of sector } AOBP}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector } AOBP = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2\theta$$

$$\text{Area of sector } AOBP = \frac{1}{2} r^2\theta$$

Example 3: Find area of the sector of a circle of radius 16cm if the angle at the centre is 60°.



Solution: Area of sector $= \frac{1}{2} r^2 \theta$

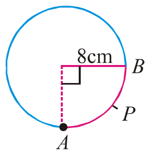
Now $\theta = 60^\circ \times \frac{\pi}{180} = \frac{\pi}{3}$ rad , $r = 16$cm

Area of sector $= \frac{1}{2}(16)^2 \left(\frac{\pi}{3}\right)$

$= \frac{1}{2}(256) \times \left(\frac{22}{7 \times 3}\right) \simeq 134.1 \text{ cm}^2$

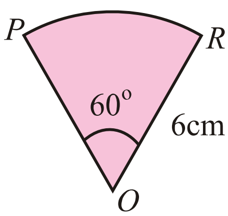
EXERCISE 7.2

1. Find θ, when:
   (i) $l = 2$cm , $r = 3.5$cm (ii) $l = 4.5$m , $r = 2.5$m
2. Find l, when:
   (i) $\theta = 180^\circ$, $r = 4.9$cm (ii) $\theta = 60^\circ 30'$, $r = 15$mm
3. Find r, when:
   (i) $l = 4$cm , $\theta = \frac{1}{4}$ radian (ii) $l = 52$cm , $\theta = 45^\circ$
4. In a circle of radius 12m, find the length of an arc which subtends a central angle $\theta = 1.5$ radian.
5. In a circle of radius 10m, find the distance travelled by a point moving on this circle if the point makes 3.5 revolution. (3.5 revolution = 7π).
6. What is the circular measure of the angle between the hands of the watch at 3 o'clock?
7. What is the length of the arc APB?

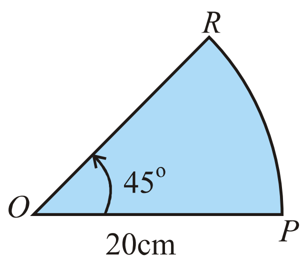


8. In a circle of radius 12cm, how long an arc subtends a central angle of 84°.
9. Find the area of the sectors OPR.

(a)

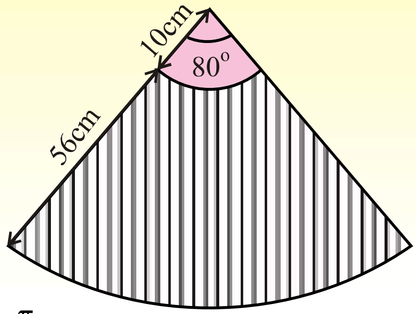


(b)



10. Find area of the sector inside a central angle of 20^{o} in a circle of radius 7m.

11. Sehar is making a skirt. Each panel of this skirt is of the shape shown shaded in the diagram. How much material (cloth) is required for each panel?



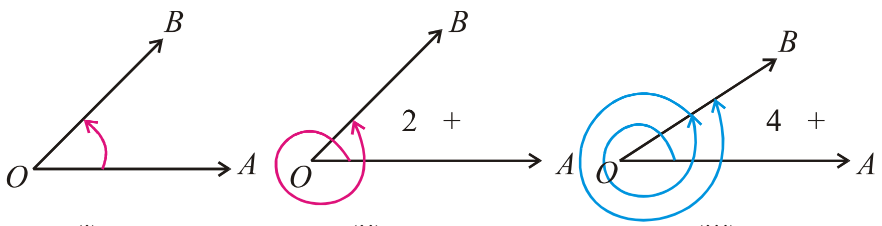
12. Find the area of the sector with central angle of $\frac{\pi}{5}$ radian in a circle of radius 10cm.

13. The area of the sector with a central angle θ in a circle of radius 2m is 10 square meter. Find θ in radians.

7.3 Trigonometric Ratios

7.3(i-a) General Angles (Coterminal angles)

An angle is indicated by a curved arrow that shows the direction of rotation from initial to the terminal side. Two or more than two angles may have the same initial and terminal sides. Consider an angle $\angle AOB$ with $\overline{OA}$ as initial side and $\overline{OB}$ as terminal side with vertex O. Let $m\angle AOB = \theta$ radian, where $0 \leq \theta \leq 2\pi$.



(i) (ii) (iii)

If the terminal side $\overline{OB}$ comes to its original position after, one, two or more than two complete revolutions in the anti-clockwise direction, then $m\angle AOB$ in above three cases will be

(i)	θ rad	after zero revolution
(ii)	$(2\pi + \theta)$ rad.	after one revolution
(iii)	$(4\pi + \theta)$ rad.	after two revolutions

Coterminal Angles: Two or more than two angles with the same initial and terminal sides are called **coterminal angles**.

It means that terminal side comes to its original position after every revolution of 2π radian in anti=clockwise or clockwise direction. In general, if θ is in degrees, then $360^{o}k + \theta$, where $k \in Z$, is an angle coterminal with θ. If angle θ is in radian measure, then $2k\pi + \theta$ where $k \in Z$, is an angle coterminal with θ.

Thus, the general angle $\theta = 2(k)\pi + \theta$, where $k \in Z$.

Example: Which of following angles are coterminal with 120^o?

$$-240^o, 480^o, \frac{14\pi}{3} \text{ and } -\frac{14\pi}{3}$$

Solution: -240^o is conterminal with 120^o as their terminal side is same

$480^o = 360^o + 120^o$, the angle 480^o terminates at 120^o after one complete revolution.

$\frac{14}{3}\pi \equiv 4\pi + \frac{2\pi}{3} = 720^o + 120^o$ the angle $\frac{14\pi}{3}$ is coterminal with 120^o.

$\frac{-14\pi}{3} = -4\pi + \frac{-2\pi}{3} = -720^o - 120^o$ so $\frac{-14\pi}{3}$ is not coterminal with 120^o.

7.3(i-b) Angle in Standard Position

A general angle is said to be in standard position if its vertex is at the origin and its initial side is directed along the positive direction of the x-axis of a rectangular coordinate system.

Because all the angles in standard position have the same initial side, the location of the terminal side is of importance. The position of the terminal side of an angle in standard position remains the same if measure of the angle is increased or decreased by a multiple of 2π.

Some standard angles are shown in the following figures:

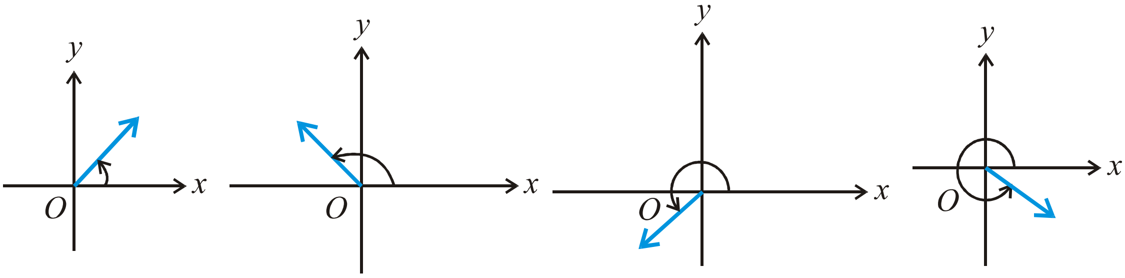


Fig. 7.3.1 (a)

Example: Locate each angle in standard position.

(i) 240^o (ii) 490^o (iii) -270^o

Solution: The angles are shown in Figure 7.3.1 (b)

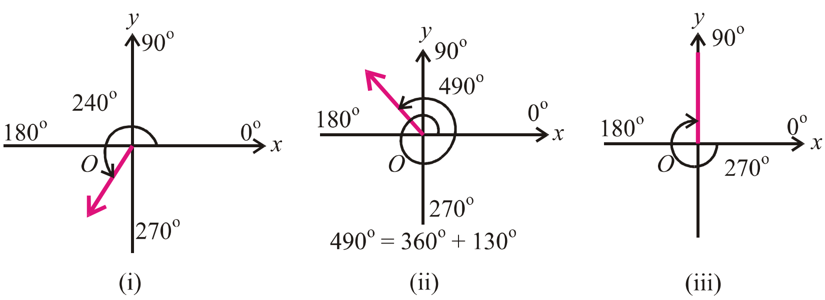
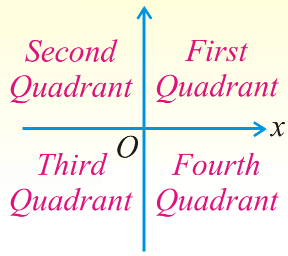


Fig. 7.3.1 (b)

7.3(ii) The Quadrants and Quadrantal Angles

The x-axis and y-axis divides the plane in four regions, called **quadrants**, when they intersect each other at right angle. The point of intersection is called **origin** and is denoted by O.



Angles between 0^o and 90^o are in the first quadrant.
Angles between 90^o and 180^o are in the second quadrant.
Angles between 180^o and 270^o are in the third quadrant.
Angles between 270^o to 360^o are in the fourth quadrant.
An angle in standard position is said to lie in a quadrant if its terminal side lies in that quadrant. Angles α, β, γ and θ lie in I, II, III and IV quadrant respectively in figure 7.3.1.

Quadrantal Angles

If the terminal side of an angle in standard position falls on x-axis or y-axis, then it is called a **quadrantal angle** *i.e.,* 90^o, 180^o, 270^o and 360^o are quadrantal angles. The quadrantal angles are shown as below:

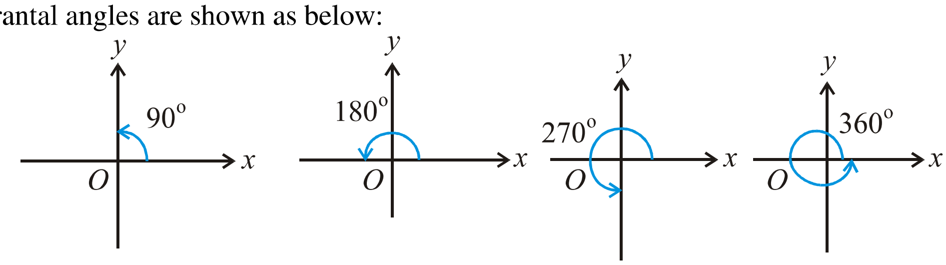


Fig. 7.3.2

7.3(iii) Trigonometric ratios and their reciprocals with the help of a unit circle.

There are six fundamental trigonometric ratios called sine, cosine, tangent, cotangent, secant and cosecant. To define these functions we use circular approach which involves the unit circle.

Let θ be a real number, which represents the radian measure of an angle in standard position. Let $P\ (x, y)$ be any point on the unit circle lying on terminal side of θ as shown in the figure.

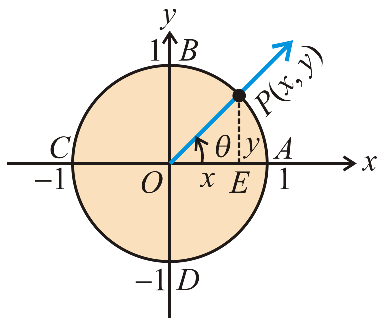


Fig. 7.3.3

We define sine of θ, written as $\sin\theta$ and cosine of θ written as $\cos\theta$, as:

$$\sin\theta = \frac{EP}{OP} = \frac{y}{1} \quad \Rightarrow \quad \sin\theta = y$$

and $$\cos\theta = \frac{OE}{OP} = \frac{x}{1} \quad \Rightarrow \quad \cos\theta = x$$

i.e., $\cos\theta$ and $\sin\theta$ are the x-coordinate and y-coordinate of the point P on the unit circle. The equations $x = \cos\theta$ and $y = \sin\theta$ are called **circular** or **trigonometric functions**.

The remaining trigonometric functions tangent, cotangent, secant and cosecant will be denoted by $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ for any real angle θ.

$\tan\theta = \frac{EP}{OE} = \frac{y}{x} \quad \Rightarrow \quad \tan\theta = \frac{y}{x} \qquad (x \neq 0)$

since $\quad y = \sin\theta$ and $x = \cos\theta \quad \Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{x}{y} \quad (y \neq 0) \quad \Rightarrow \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{x}\,(x \neq 0) \quad$ and $\quad \mathrm{cosec}\,\theta = \frac{1}{y} \quad (y \neq 0)$

$= \frac{1}{\cos\theta} \qquad\qquad = \frac{1}{\sin\theta}$

Reciprocal Identities

$\sin\theta = \frac{1}{\mathrm{cosec}\,\theta}$	or	$\mathrm{cosec}\,\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	or	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	or	$\cot\theta = \frac{1}{\tan\theta}$

Example 2: Find the value of the trigonometric ratios at θ if point (3, 4) is on the terminal sides of θ.

Solution: We have $x = 3$ and $y = 4$

We shall also need value of r, which is found by using the fact that

$r = \sqrt{x^2 + y^2} \quad ; \quad r = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$ where $r = |OP|$

Thus $\quad \sin\theta = \frac{y}{r} = \frac{4}{5} \quad ; \quad \mathrm{cosec}\,\theta = \frac{5}{4}$

$\cos\theta = \frac{x}{r} = \frac{3}{5} \quad ; \quad \sec\theta = \frac{5}{3}$

$\tan\theta = \frac{y}{x} = \frac{4}{3} \quad ; \quad \cot\theta = \frac{3}{4}$

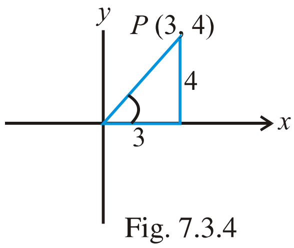


Fig. 7.3.4

7.3(iv) The values of trigonometric ratio for 45°, 30°, 60°.

Consider a right triangle ABC with $m\angle C = 90°$. The sides opposite to the vertices A, B and C are denoted by a, b and c respectively.

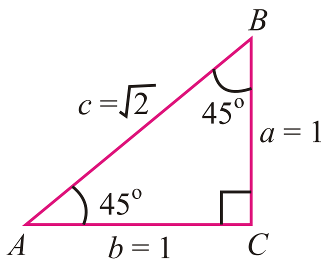


Fig. 7.3.5

Case I When $m\angle A = 45°$, where $45° = \frac{\pi}{4}$ radian. Since the sum of angles in a triangle is 180°, so $m\angle B = 45°$.

As values of trigonometric functions depend on the size of the angle only and not on the size of triangle. For convenience, we take $a = b = 1$. In this case the triangle is isosceles right triangle.

By Pythagorean theorem.

$$c^2 = a^2 + b^2 \quad \Rightarrow \quad c^2 = (1)^2 + (1)^2 = 2$$

$$c^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}$$

From this triangle, we have

$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{a}{c} = \frac{1}{\sqrt{2}} \quad ; \quad \operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$$

$$\cos 45^\circ = \cos\frac{\pi}{4} = \frac{b}{c} = \frac{1}{\sqrt{2}} \quad ; \quad \sec 45^\circ = \frac{1}{\cos 45} = \sqrt{2}$$

$$\tan 45^\circ = \tan\frac{\pi}{4} = \frac{a}{b} = \frac{1}{1} = 1 \; ; \quad \cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

Case II When $m\angle A = 30^\circ$ or $m\angle A = 60^\circ$

Consider an equilateral triangle with sides $a = b = c = 2$ for convenience. Since the angles in an equilateral triangle are equal and their sum is 180°, each angle has measure 60°. Bisecting an angle in the triangle, we obtain two right triangles with 30° and 60° angles. The height $|AD|$ of these triangles may be found by Pythagorean theorem, *i.e.*,

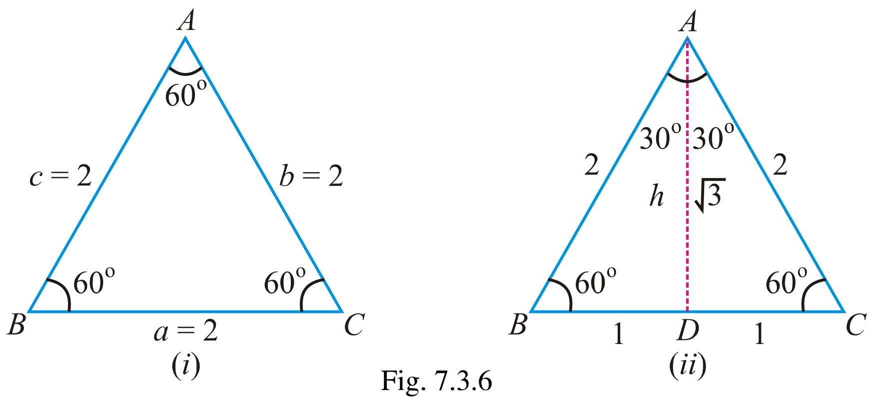


Fig. 7.3.6

$$(AD)^2 + (BD)^2 = (AB)^2 \quad \Rightarrow \quad (AD)^2 = (AB)^2 - (BD)^2$$

$$h^2 = (2)^2 - (1)^2 = 3$$

$$\Rightarrow \quad h = \sqrt{3}$$

$\therefore$ Using triangle ADB with $m\angle A = 30^\circ$, we have

$$\sin 30^\circ = \sin\frac{\pi}{6} = \frac{BD}{AB} = \frac{1}{2} \qquad \operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\cos 30^\circ = \cos\frac{\pi}{6} = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \tan\frac{\pi}{6} = \frac{BD}{AD} = \frac{1}{\sqrt{3}} \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

Now using triangle ABD with $m\angle B = 60^\circ$.

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \operatorname{cosec} 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{1}{2} \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}}{1} \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$$

7.3(v) Signs of trigonometric ratios in different quadrants

In case of trigonometric ratios like $\sin\theta$, $\cos\theta$ and $\tan\theta$ if θ is not a quadrantal angle, then θ will lie in a particular quadrant. Since $r = \sqrt{x^2 + y^2}$ is always +*ve*, the signs of ratios can be found if the quadrant of θ is known.

(*i*) If θ lies in first quadrant, then a point P (x, y) on its terminal side has x and y co-ordinate positive.
Therefore, all trigonometric functions are positive in quadrant I.

(*ii*) If θ lies in 2nd quadrant, then point $P(x, y)$ on its terminal side has negative x-coordinate and positive y-coordinate.

$\therefore$ $\sin\theta = \frac{y}{r}$ is +*ve* or > 0 $\quad\cos\theta = \frac{x}{r}$ is −*ve* or < 0 and $\tan\theta = \frac{y}{x}$ is −*ve* or < 0

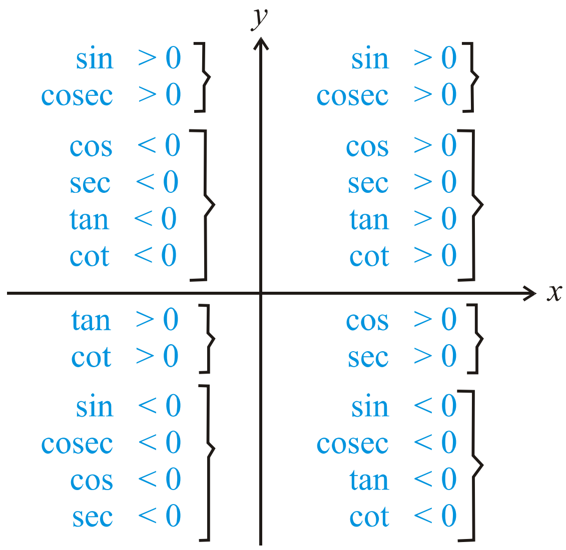
(*iii*) When θ lies in third quadrant, then a point P (x, y) on its terminal side has negative x-coordinate and negative y-coordinate.

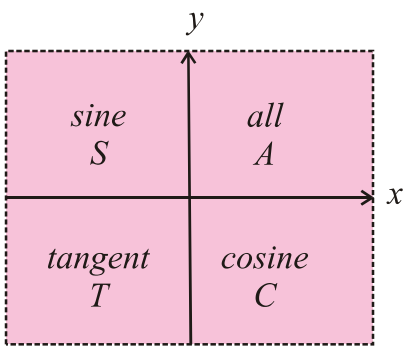
$\therefore$ $\sin\theta = \frac{y}{r}$ is −*ve* or < 0, $\cos\theta = \frac{x}{r}$ is −*ve* or < 0 and $\tan\theta = \frac{y}{x}$ is +*ve* or > 0

(*iv*) When θ lies in fourth quadrant, then the point P (x, y) on the terminal side of θ has positive x-coordinate and negative y-coordinate.

$\therefore$ $\sin\theta = \frac{y}{r}$ is −*ve* or < 0, $\cos\theta = \frac{x}{r}$ is +*ve* or > 0 and $\tan\theta = \frac{y}{x}$ is −*ve* or < 0

The signs of all trigonometric functions are summarized as below.





Read 'ASTC' as
'Add Sugar To Coffee'

7.3(vi) Values of remaining trigonometric ratios if one trigonometric ratio is given

The method is illustrated by the following examples:

Example 1: If $\sin\theta = \frac{-3}{4}$ and $\cos\theta = \frac{\sqrt{7}}{4}$, then find the values of $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\text{cosec}\,\theta$.

Solution: Applying the identities that express the remaining trigonometric functions in terms of sine and cosine, we have

$$\because \quad \sin\theta = \frac{-3}{4} \quad \therefore \quad \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{-3}{4}} = \frac{-4}{3} \quad \Rightarrow \quad \text{cosec}\,\theta = \frac{-4}{3}$$

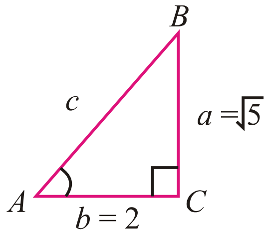
$$\because \quad \cos\theta = \frac{\sqrt{7}}{4} \quad \therefore \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{7}}{4}} \quad \Rightarrow \quad \sec\theta = \frac{4}{\sqrt{7}}$$

$$\text{Now} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{-3}{4}}{\frac{\sqrt{7}}{4}} = \frac{-3}{\sqrt{7}} \quad \Rightarrow \quad \tan\theta = \frac{-3}{\sqrt{7}}$$

$$\text{and} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{-\sqrt{7}}{3}$$

Example 2: If $\tan\theta = \frac{\sqrt{5}}{2}$, then find the values of other trigonometric ratios at θ.

Solution: In any right triangle ABC,



$$\tan\theta = \frac{\sqrt{5}}{2} = \frac{a}{b} \quad \Rightarrow \quad a = \sqrt{5},\ b = 2$$

Now by Pythagorean theorem

$$a^2 + b^2 = c^2 \quad \Rightarrow \quad (\sqrt{5})^2 + (2)^2 = c^2$$

$$c^2 = 5 + 4 = 9 \quad \Rightarrow \quad c = \pm 3 \quad \text{or} \quad c = 3$$

$$\because \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\therefore \quad \cot\theta = \frac{1}{\frac{\sqrt{5}}{2}} \quad \Rightarrow \quad \cot\theta = \frac{2}{\sqrt{5}}$$

$$\sin\theta = \frac{a}{c} = \frac{\sqrt{5}}{3}, \quad \text{cosec}\,\theta = \frac{1}{\sin\theta} \quad \Rightarrow \quad \text{cosec}\,\theta = \frac{1}{\frac{\sqrt{5}}{3}} \quad \therefore \quad \text{cosec}\,\theta = \frac{3}{\sqrt{5}}$$

$$\text{Also} \quad \cos\theta = \frac{b}{c} = \frac{2}{3}, \quad \sec\theta = \frac{1}{\cos\theta} \quad \Rightarrow \quad \sec\theta = \frac{1}{\frac{2}{3}} \quad \therefore \quad \sec\theta = \frac{3}{2}$$

7.3(vii) Calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°

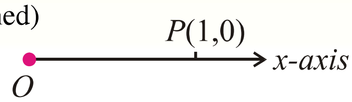
We have discussed quadrantal angles in section 7.3.2. An angle θ is called a quadrantal angle if its terminal side lies on the x-axis or on the y-axis.

Case I When $\theta = 0^\circ$

The point (1, 0) lies on the terminal side of angle θ°. We may consider the point on the unit circle on the terminal side of the angle.

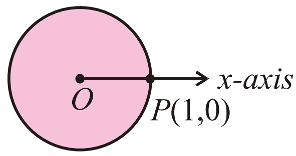
$$P(1, 0) \quad \Rightarrow \quad x = 1 \text{ and } y = 0 \quad \text{so} \quad r = \sqrt{x^2 + y^2} = \sqrt{1 + 0} = 1$$

$$\therefore \quad \sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0 \quad , \quad \operatorname{cosec} 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty \quad \text{(undefined)}$$



$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1 \quad , \quad \sec 0^\circ = \frac{1}{\cos 0^\circ} = 1$$

$$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0 \quad , \quad \cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = \infty \quad \text{(undefined)}$$

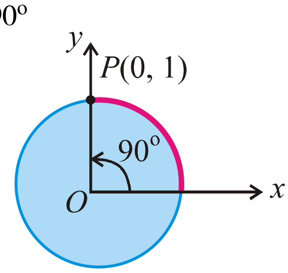


Case II When $\theta = 90^\circ$

The point P (0, 1) lies on the terminal side of angle 90°

Here $\quad x = 0$ and $y = 1 \quad \Rightarrow \quad r = \sqrt{0^2 + (1)^2} = 1$

$$\therefore \quad \sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1$$



i.e., $\sin 90^\circ = 1$ and $\operatorname{cosec} 90^\circ = \frac{r}{y} = 1$

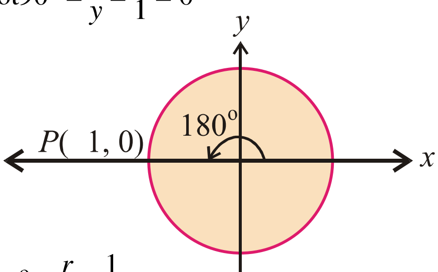
Using reciprocal identities, we have

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0 \quad ; \quad \sec 90^\circ = \frac{r}{x} = \frac{1}{0} = \infty \quad \text{(undefined)}$$

$$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \infty \text{ (undefined)} \, , \quad \cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$$

Case III When $\theta = 180^\circ$

The point P (−1, 0) lies on x'-axis or on terminal side of angle 180°



Here $\quad x = -1 \quad$ and $\quad y = 0$

$$\Rightarrow \quad r = \sqrt{x^2 + y^2} = 1$$

$$\therefore \quad \sin 180^\circ = \frac{y}{r} = \frac{0}{1} = 0 \quad ; \quad \operatorname{cosec} 180^\circ = \frac{r}{y} = \frac{1}{0} = \infty \quad \text{(undefined)}$$

$$\cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1 \quad ; \quad \sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0 \; ; \quad \cot 180^\circ = \frac{x}{y} = \frac{-1}{0} = \infty \quad \text{(undefined)}$$

Case IV When $\theta = 270^\circ$ and the point P (0, −1) lies on y'-axis or on the terminal side of angle 270°.

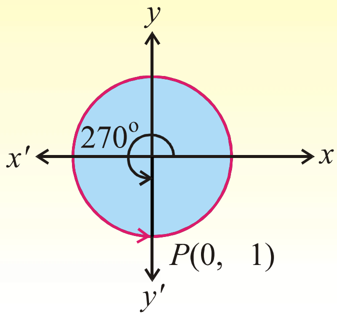
The point P (0, –1) shows that $x = 0$ and $y = -1$

So $\quad r = \sqrt{(0)^2 + (-1)^2} = 1$

$\therefore \; \sin 270^o = \frac{y}{r} = \frac{-1}{1} = -1 \quad ; \quad \text{cosec} 270^o = \frac{r}{y} = \frac{1}{-1} = -1$

$\cos 270^o = \frac{x}{r} = \frac{0}{1} = 0 \quad ; \quad \sec 270^o = \frac{r}{x} = \frac{1}{0} = \infty$

$\tan 270^o = \frac{y}{x} = \frac{-1}{0} = -\infty \quad ; \quad \cot 270^o = \frac{x}{y} = \frac{0}{-1} = 0$



Case V When $\theta^o = 360^o$

Now the point P (1, 0) lies once again on x-axis

We know that $\theta + 2k\pi = \theta$ where $k \in Z$.

Now $\quad \theta = 360^o = 0^o + (360^o)\,1 = 0^o$ where $k = 1$

So $\quad \sin 360^o = \sin 0^o = 0 \quad ; \quad \text{cosec} 360^o = \frac{1}{\sin 360^o} = \frac{1}{\sin 0^o} = \frac{1}{0} = \infty$ (undefined)

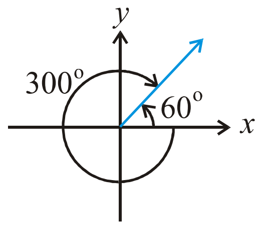
$\cos 360^o = \cos 0^o = 1 \quad ; \quad \sec 360^o = \frac{1}{\cos 0^o} = \frac{1}{1} = 1,$

$\tan 360^o = \tan 0^o = 0 \quad ; \quad \cot 360^o = \frac{1}{\tan 0^o} = \frac{1}{0} = \infty$ (undefined)

Example: Find each of the following without using table or calculator:

(i) $\cos 540^o$ (ii) $\sin 315^o$ (iii) $\sec(-300^o)$

Solution: We know that $2k\pi + \theta = \theta$, where $k \in Z$.



(i) $\quad 540^o = (360^o + 180^o) = 2(1)\pi + 180^o$

$\cos 540^o = \cos(2\pi + \pi) = \cos\pi = -1$

(ii) $\quad \sec 315^o = \sin(360^o - 45^o) = \sin\left(2\pi - \frac{\pi}{4}\right)$

$= \sin\left(\frac{-\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

(iii) $\quad \sec(-300^o) = \sec(-360^o + 60^o)$

$= \sec(2(-1)\pi + 60^o)$

$= \sec 60^o = \frac{1}{\cos 60^o} = \frac{1}{\frac{1}{2}} = 2$

EXERCISE 7.3

1. Locate each of the following angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle coterminal with each given angle.

   (i) 170^o (ii) 780^o (iii) -100^o (iv) -500^o

2. Identify the closest quadrantal angles between which the following angles lies.

   (i) 156^o (ii) 318^o (iii) 572^o (iv) -330^o

3. Write the closest quadrantal angles between which the angle lies. Write your answer in radian measure.

   (i) $\frac{\pi}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{-\pi}{4}$ (iv) $\frac{-3\pi}{4}$

4. In which quadrant θ lie when

   (i) $\sin\theta > 0$, $\tan\theta < 0$ (ii) $\cos\theta < 0$, $\sin\theta < 0$

   (iii) $\sec\theta > 0$, $\sin\theta < 0$ (iv) $\cos\theta < 0$, $\tan\theta < 0$

   (v) $\operatorname{cosec}\theta > 0$, $\cos\theta > 0$ (vi) $\sin\theta < 0$, $\sec\theta < 0$

5. Fill in the blanks.

   (i) $\cos(-150^o)$ = ... $\cos 150^o$ (ii) $\sin(-310^o)$ = ... $\sin 310^o$

   (iii) $\tan(-210^o)$ = ... $\tan 210^o$ (iv) $\cot(-45^o)$ = ... $\cot 45^o$

   (v) $\sec(-60^o)$ = ... $\sec 60^o$ (vi) $\operatorname{cosec}(-137^o)$ = ... $\operatorname{cosec} 137^o$

6. The given point P lies on the terminal side of θ. Find quadrant of θ and all six trigonometric ratios.

   (i) $(-2, 3)$ (ii) $(-3, -4)$ (iii) $(\sqrt{2}, 1)$

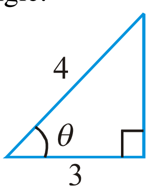
7. If $\cos\theta = \frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, find the values of remaining trigonometric functions.

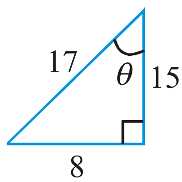
8. If $\tan\theta = \frac{4}{3}$ and $\sin\theta < 0$, find the values of other trigonometric functions at θ.

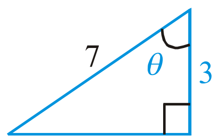
9. If $\sin\theta = \frac{-1}{\sqrt{2}}$ and terminal side of the angle is not in quadrant III, find the values of $\tan\theta$, $\sec\theta$, and $\operatorname{cosec}\theta$.

10. If $\operatorname{cosec}\theta = \frac{13}{12}$ and $\sec\theta > 0$, find the remaining trigonometric functions.

11. Find the values of trigonometric functions at the indicated angle θ in the right triangle.

   (i) 

   (ii) 

   (iii) 

12. Find the values of the trigonometric functions. Do not use trigonometric tables or calculator.

(i) $\tan 30^\circ$ (ii) $\tan 330^\circ$ (iii) $\sec 330^\circ$

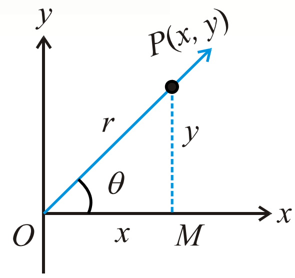
(iv) $\cot \frac{\pi}{4}$ (v) $\cos \frac{2\pi}{3}$ (vi) $\operatorname{cosec} \frac{2\pi}{3}$

(vii) $\cos(-450^\circ)$ (viii) $\tan(-9\pi)$ (ix) $\cos\left(\frac{-5\pi}{6}\right)$

(x) $\sin \frac{7\pi}{6}$ (xi) $\cot \frac{7\pi}{6}$ (xii) $\cos 225^\circ$

7.4 Trigonometric Identities

We have discussed trigonometric functions (ratios) and their reciprocals in section 7.3. Consider an angle $\angle MOP = \theta$ radian in standard position. Let point $P(x, y)$ be on the terminal side of the angle. By Pythagorean theorem, we have from right triangle OMP.



$$OM^2 + MP^2 = OP^2$$

$$x^2 + y^2 = r^2 \quad \ldots\ldots \text{(i)}$$

$$\because \sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}$$

Dividing both sides by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\Rightarrow \quad \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\Rightarrow \quad (\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\therefore \quad \cos^2\theta + \sin^2\theta = 1 \qquad (1)$$

Dividing (i) by x^2, we have

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$\Rightarrow \quad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \qquad \because \tan\theta = \frac{y}{x} \text{ and } \sec\theta = \frac{r}{x}$$

$$\Rightarrow \quad 1 + (\tan\theta)^2 = (\sec\theta)^2$$

$$\therefore \quad 1 + \tan^2\theta = \sec^2\theta \quad \text{or} \quad \sec^2\theta - \tan^2\theta = 1 \qquad (2)$$

Again dividing both sides of (i) by y^2, we get

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2 \qquad \because \cot\theta = \frac{x}{y} \text{ and } \operatorname{cosec}\theta = \frac{r}{y}$$

$$\Rightarrow \quad (\cot\theta)^2 + 1 = (\operatorname{cosec}\theta)^2$$

$\therefore \quad 1 + \cot^2\theta = \text{cosec}^2\theta$ or $\text{cosec}^2\theta - \cot^2\theta = 1 \qquad (3)$

The identities (1), (2) and (3) are also known as **Pythagorean Identities.**

The fundamental identities are used to simplify expressions involving trigonometric functions.

Example 1: Verify that $\cot\theta\sec\theta = \text{cosec}\,\theta$

Solution: Expressing left hand side in terms of sine and cosine, we have

$$\text{L.H.S} = \cot\theta\sec\theta = \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta}$$

$$= \frac{1}{\sin\theta} = \text{cosec}\,\theta$$

$$= \text{R.H.S}$$

Example 2: Verify that $\tan^4\theta + \tan^2\theta = \tan^2\theta\sec^2\theta$

Solution: $\text{L.H.S} = \tan^4\theta + \tan^2\theta = \tan^2\theta(\tan^2\theta + 1) \qquad \because \tan^2\theta + 1 = \sec^2\theta$

$$= \tan^2\theta\sec^2\theta$$

$$= \text{R.H.S}$$

Example 3: Show that $\dfrac{\cot^2\alpha}{\text{cosec}\,\alpha - 1} = \text{cosec}\,\alpha + 1$

Solution: $\text{L.H.S} = \dfrac{\cot^2\alpha}{\text{cosec}\,\alpha - 1} \qquad \left(\begin{array}{l}\because \text{cosec}^2\theta - \cot^2\theta = 1 \\ \cot^2\theta = \text{cosec}^2\theta - 1\end{array}\right)$

$$= \frac{(\text{cosec}^2\alpha - 1)}{\text{cosec}\,\alpha - 1} = \frac{(\text{cosec}\,\alpha - 1)(\text{cosec}\,\alpha + 1)}{(\text{cosec}\,\alpha - 1)} = \text{cosec}\,\alpha + 1 = \text{R.H.S}$$

Example 4: Express the trigonometric functions in terms of $\tan\theta$.

Solution: By using reciprocal identity, we can express $\cot\theta$ in terms of $\tan\theta$.

i.e., $\quad \cot\theta = \dfrac{1}{\tan\theta}$

By solving the identity $\quad 1 + \tan^2\theta = \sec^2\theta$

We have expressed $\sec\theta$ in terms of $\tan\theta$.

$$\sec\theta = \pm\sqrt{\tan^2\theta + 1}$$

Note: We can express all the trigonometric functions in terms of one trigonometric function.

$$\because \quad \cos\theta = \frac{1}{\sec\theta} \quad \Rightarrow \quad \cos\theta = \frac{1}{\pm\sqrt{\tan^2\theta + 1}}$$

Because $\sin\theta = \tan\theta\cos\theta$, we have

$$\sin\theta = \tan\theta\left(\frac{1}{\pm\sqrt{\tan^2\theta + 1}}\right) = \frac{\tan\theta}{\pm\sqrt{\tan^2\theta + 1}}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{\pm\sqrt{\tan^2\theta + 1}}{\tan\theta}$$

EXERCISE 7.4

In Problems 1–6, simplify each expression to a single trigonometric function.

1. $\dfrac{\sin^2 x}{\cos^2 x}$

2. $\tan x \sin x \sec x$

3. $\dfrac{\tan x}{\sec x}$

4. $1 - \cos^2 x$

5. $\sec^2 x - 1$

6. $\sin^2 x \,.\, \cot^2 x.$

In problems 7-24, verify the identities.

7. $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$

8. $\dfrac{\sin\theta + \cos\theta}{\cos\theta} = 1 + \tan\theta$

9. $(\tan\theta + \cot\theta)\tan\theta = \sec^2\theta$

10. $(\cot\theta + \operatorname{cosec}\theta)(\tan\theta - \sin\theta) = \sec\theta - \cos\theta$

11. $\dfrac{\sin\theta + \cos\theta}{\tan^2\theta - 1} = \dfrac{\cos^2\theta}{\sin\theta - \cos\theta}$

12. $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta = \operatorname{cosec}\theta$

13. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

14. $\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta$

15. $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$

16. $(\tan\theta + \cot\theta)(\cos\theta + \sin\theta) = \sec\theta + \operatorname{cosec}\theta$

17. $\sin\theta(\tan\theta + \cot\theta) = \sec\theta$

18. $\dfrac{1 + \cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1 + \cos\theta} = 2\operatorname{cosec}\theta$

19. $\dfrac{1}{1 - \cos\theta} + \dfrac{1}{1 + \cos\theta} = 2\operatorname{cosec}^2\theta$

20. $\dfrac{1 + \sin\theta}{1 - \sin\theta} - \dfrac{1 - \sin\theta}{1 + \sin\theta} = 4\tan\theta\sec\theta$

21. $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

22. $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)$

23. $\sqrt{\dfrac{1 + \cos\theta}{1 - \cos\theta}} = \dfrac{\sin\theta}{1 - \cos\theta}$

24. $\sqrt{\dfrac{\sec\theta + 1}{\sec\theta - 1}} = \dfrac{\sec\theta + 1}{\tan\theta}$

7.5 Angle of Elevation and Angle of Depression

One of the objects of trigonometry is to find the distances between points or the heights of objects, without actually measuring these distances or heights.

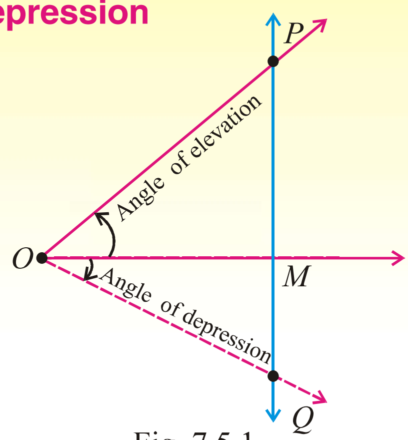


Fig. 7.5.1

Angle of elevation: Suppose O, P and Q are three points, P being at a higher level of O and Q being at lower level than O. Let a horizontal line drawn through O meet in M, the vertical line drawn through P and Q.

The angle MOP is called the **angle of elevation** of point P as seen from O. For looking at Q below the horizontal line we have to lower our eyes and $\angle MOQ$ is called the **angle of depression.**

We measure an angle of elevation from a horizontal line up to an object or an angle of depression from a horizontal line down to an object, see figure 7.5.2.

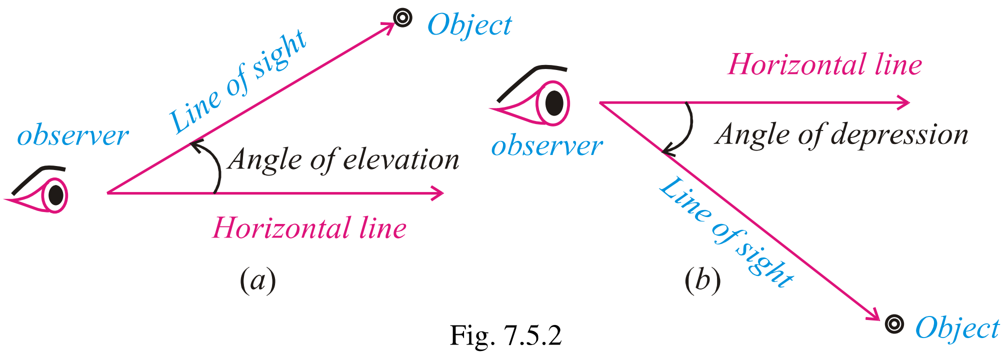
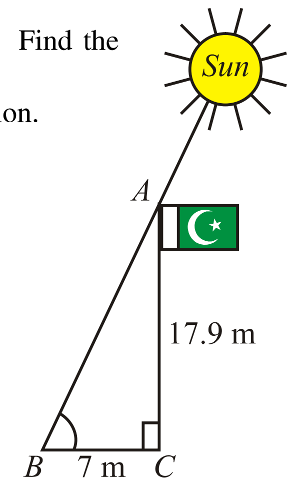


Fig. 7.5.2

7.5(i) Find angle of elevation and angle of depression:

For finding distances, heights and angles by the use of trigonometric functions, consider the following examples:



Example 1: A flagpole 17.9 meter high casts a 7 meter shadow Find the angle of elevation of the sun.

Solution: From the figure, we observe that α is the angle of elevation.

Using the fact that

$$\tan \alpha = \frac{AC}{BC} = \frac{17.9}{7} \approx 2.55714$$

Solving for α gives us

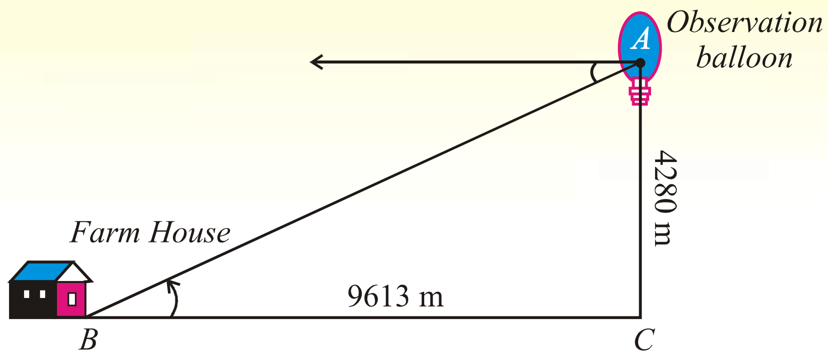
$$\alpha \approx \tan^{-1}(2.55714)$$

$$\approx (68.6666)^{o} \approx 68^{o}\ 40'$$

$$\Rightarrow \quad \alpha \approx 68^{o}\ 40'$$

Example 2: An observation balloon is 4280 meter above the ground and 9613 meter away from a farmhouse. Find the angle of depression of the farmhouse as observed from the observation balloon.

Solution:



For problems of this type the angle of elevation of *A* from *B* is considered equal to the angle of depression of *B* from *A*, as shown in the diagram.

$$\tan\alpha = \frac{AC}{BC} = \frac{4280}{9613} = 0.44523$$

$$\alpha = \tan^{-1}(0.44523) = 24^{o}$$

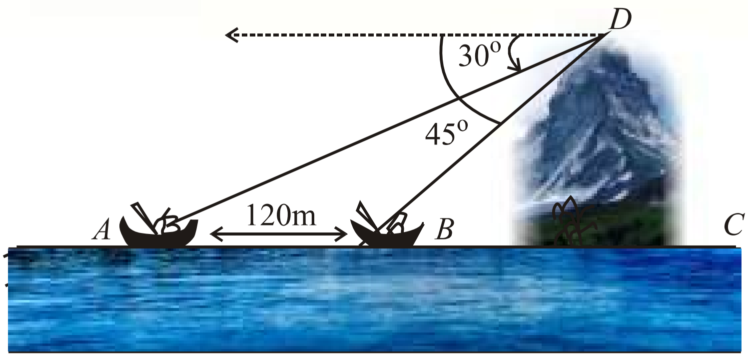
So, angle of depression is 24^{o}.

EXERCISE 7.5

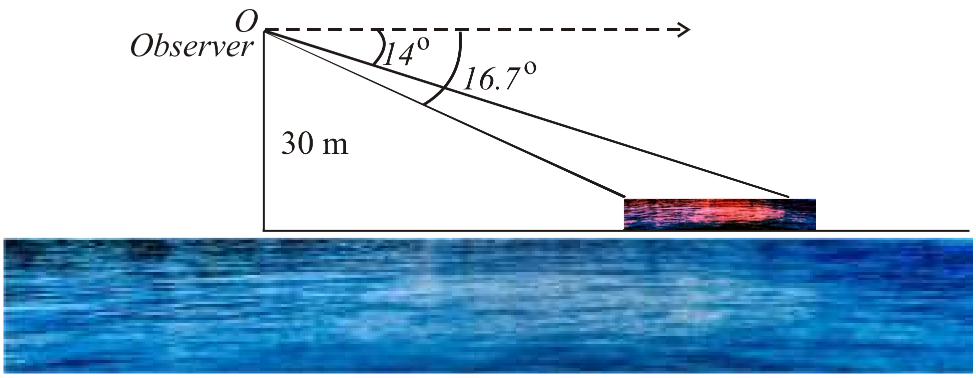
1. Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow.
2. A tree casts a 40 meter shadow when the angle of elevation of the sun is 25^{o}. Find the height of the tree.
3. A 20 feet long ladder is leaning against a wall. The bottom of the ladder is 5 feet from the base of the wall. Find the acute angle (angle of elevation) the ladder makes with the ground.
4. The base of a rectangle is 25 feet and the height of the rectangle is 13 feet. Find the angle that the diagonal of the rectangle makes with the base.
5. A rocket is launched and climbs at a constant angle of 80^{o}. Find the altitude of the rocket after it travels 5000 meter.
6. An aeroplane pilot flying at an altitude of 4000m wishes to make an approach to an airport at an angle of 50^{o} with the horizontal. How far from the airport will the plane be when the pilot begins to descend?
7. A guy wire (supporting wire) runs from the middle of a utility pole to the ground. The wire makes an angle of 78.2^{o} with the ground and touches the ground 3 meters from the base of the pole. Find the height of the pole.
8. A road is inclined at an angle 5.7^{o}. Suppose that we drive 2 miles up this road starting from sea level. How high above sea level are we?
9. A television antenna of 8 feet height is located on the top of a house. From a point on the ground the angle of elevation to the top of the house is 17^{o} and the angle of elevation to the top of the antenna is 21.8^{o}. Find the height of the house.

10. From an observation point, the angles of depression of two boats in line with this point are found to be 30^{o} and 45^{o}. Find the distance between the two boats if the point of observation is 4000 feet high.

11. Two ships, which are in line with the base of a vertical cliff, are 120 meters apart. The angles of depression from the top of the cliff to the ships are 30^{o} and 45^{o}, as shown in the diagram.
   (a) Calculate the distance *BC*
   (b) Calculate the height *CD*, of the cliff.



12. Suppose that we are standing on a bridge 30 feet above a river watching a log (piece of wood) floating towards us. If the angle with the horizontal to the front of the log is 16.7^{o} and angle with the horizontal to the back of the log is 14^{o}, how long is the log?



MISCELLANEOUS EXERCISE - 7

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The union of two non-collinear rays, which have common end point is called
(*a*) an angle (*b*) a degree (*c*) a minute (*d*) a radian

(ii) The system of measurement in which the angle is measured in radians is called
(*a*) CGS system (*b*) sexagesimal system
(*c*) MKS system (*d*) circular system

(iii) $20^\circ =$
(*a*) $360'$ (*b*) $630'$ (*c*) $1200'$ (*d*) $3600'$

(iv) $\frac{3\pi}{4}$ radians =
(*a*) 115° (*b*) 135° (*c*) 150° (*d*) 30°

(v) If $\tan\theta = \sqrt{3}$, then θ is equal to
(*a*) 90° (*b*) 45° (*c*) 60° (*d*) 30°

(vi) $\sec^2\theta =$
(*a*) $1 - \sin^2\theta$ (*b*) $1 + \tan^2\theta$ (*c*) $1 + \cos^2\theta$ (*d*) $1 - \tan^2\theta$

(vii) $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} =$
(*a*) $2\sec^2\theta$ (*b*) $2\cos^2\theta$ (*c*) $\sec^2\theta$ (*d*) $\cos\theta$

(viii) $\frac{1}{2}\operatorname{cosec}45^\circ =$
(*a*) $\frac{1}{2\sqrt{2}}$ (*b*) $\frac{1}{\sqrt{2}}$ (*c*) $\sqrt{2}$ (*d*) $\frac{\sqrt{3}}{2}$

(ix) $\sec\theta\cot\theta =$
(*a*) $\sin\theta$ (*b*) $\frac{1}{\cos\theta}$ (*c*) $\frac{1}{\sin\theta}$ (*d*) $\frac{\sin\theta}{\cos\theta}$

(x) $\operatorname{cosec}^2\theta - \cot^2\theta =$
(*a*) -1 (*b*) 1 (*c*) 0 (*d*) $\tan\theta$

2. Write short answers of the following questions.

(i) Define an angle.
(ii) What is the sexagesimal system of measurement of angles?
(iii) How many minutes are there in two right angles?
(iv) Define radian measure of an angle.
(v) Convert $\frac{\pi}{4}$ radians to degree measure.
(vi) Convert 15° to radians.
(vii) What is radian measure of the central angle of an arc 50m long on the circle of radius 25m?
(viii) Find r when $l = 56\ cm$ and $\theta = 45^\circ$
(ix) Find $\tan\theta$ when $\cos\theta = \frac{9}{41}$ and terminal side of the angle θ is in fourth quadrant.
(x) Prove that $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

3. Fill in the blanks

(i) π radians = __________degree.

(ii) The terminal side of angle 235° lies in __________quadrant.

(iii) Terminal side of the angle –30° lies in __________quadrant.

(iv) Area of a circular sector is __________.

(v) If $r = 2$ cm and $\theta = 3$ radian, then area of the circular sector is __________.

(vi) The general form of the angle 480° is __________.

(vii) If $\sin\theta = \frac{1}{2}$, then $\theta =$ __________.

(viii) If $\theta = 300^\circ$, then $\sec(-300)^\circ =$ __________.

(ix) $1 + \cot^2\theta =$ __________.

(x) $\sec\theta - \tan\theta =$ __________.

SUMMARY

- If we divide the circumference of a circle into 360 equal arcs, then the angle subtended at the centre of the circle by one arc is called one **degree** and is denoted by 1°.
- The angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle, is called one **radian.**
- **Relationship between radian and degree measure**

  $1^\circ = \frac{\pi}{180}$ radians , ≈ 0.0175 radians and 1 radian = $\left(\frac{180}{\pi}\right)^\circ$, ≈ 57.295 degrees
- Relation between central angle and arc length of a circle: $l = r\theta$
- Area of a circular sector, $A = \frac{1}{2}r^2\theta$
- Two or more than two angles with the same initial and terminal sides are called **coterminal** angles.
- An angle is called a **quadrantal angle,** if its terminal side lies on the x-axis or y-axis.
- A general angle is said to be in **standard position** if its vertex is at the origin and its initial side is directed along the positive direction of the x-axis of a rectangular coordinate system.
- There are six fundamental **trigonometric ratios** (functions) known as sine, cosine, tangent, cotangent, secant and cosecant.
- **Trigonometric Identities:**
  (a) $\cos^2\theta + \sin^2\theta = 1$
  (b) $1 + \tan^2\theta = \sec^2\theta$
  (c) $1 + \cot^2\theta = \text{cosec}^2\theta$