Unit-8

PROJECTION OF A SIDE OF A TRIANGLE

In this unit, students will learn how to

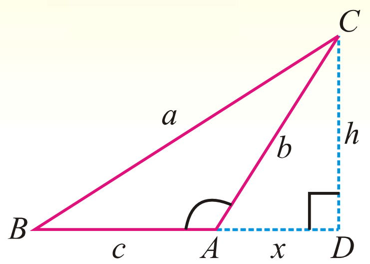
Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- *In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.*
- *In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.*
- *In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).*

THEOREM 1

8.1(*i*) In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

Given: ABC is a triangle having an obtuse angle BAC at A. Draw $\overline{CD}$ perpendicular on $\overline{BA}$ produced. So that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced.



Take $m\overline{BC} = a$, $m\overline{CA} = b$, $m\overline{AB} = c$, $m\overline{AD} = x$ and $m\overline{CD} = h$.

To prove: $(BC)^2 = (AC)^2 + (AB)^2 + 2(m\overline{AB})\,(m\overline{AD})$

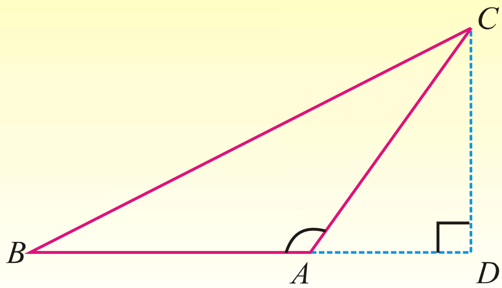
i.e., $a^2 = b^2 + c^2 + 2cx$

Proof:

Statements	Reasons
In $\angle rt\Delta\ CDA$,	
$m\angle CDA = 90^o$	Given
$\therefore$ $(AC)^2 = (AD)^2 + (CD)^2$	Pythagoras Theorem
or $b^2 = x^2 + h^2$ (i)	
In $\angle rt\Delta\ CDB$,	
$m\angle CDB = 90^o$	Given
$\therefore$ $(BC)^2 = (BD)^2 + (CD)^2$	Pythagoras Theorem
or $a^2 = (c + x)^2 + h^2$	$\overline{BD} = \overline{BA} + \overline{AD}$
$= c^2 + 2cx + x^2 + h^2$ (ii)	
Hence $a^2 = c^2 + 2cx + b^2$	Using (i) and (ii)
i.e., $a^2 = b^2 + c^2 + 2cx$	
or $(BC)^2 = (AC)^2 + (AB)^2 + 2(m\overline{AB})\,(m\overline{AD})$	

Example: In a ΔABC with obtuse angle at A, if $\overline{CD}$ is an altitude on $\overline{BA}$ produced and $m\overline{AC} = m\overline{AB}$

Then prove that $(BC)^2 = 2(AB)(BD)$



Given: In a ΔABC, $m\angle A$ is obtuse $m\overline{AC} = m\overline{AB}$ and $\overline{CD}$ being altitude on $\overline{BA}$ produced.

To prove: $(BC)^2 = 2(m\overline{AB})(m\overline{BD})$

Proof: In a ΔABC, having obtuse angle BAC at A.

Statements	Reasons
$(BC)^2 = (BA)^2 + (AC)^2 + 2(m\overline{BA})(m\overline{AD})$	By Theorem 1
$= (AB)^2 + (AB)^2 + 2(m\overline{AB})(m\overline{AD})$	Given
$= 2(AB)^2 + 2(m\overline{AB})(m\overline{AD})$	
$(BC)^2 = 2m\overline{AB}(m\overline{AB} + m\overline{AD})$	On the line segment BD,
$= 2m\overline{AB}\ (m\overline{BD})$	Point A is between B and D

EXERCISE 8.1

1. Given $m\overline{AC} = 1\text{cm}$, $m\overline{BC} = 2\text{cm}$, $m\angle C = 120^o$.

   Compute the length AB and the area of ΔABC.

   Hint: $(AB)^2 = (AC)^2 + (BC)^2 + 2mAC.\ mCD$

   where $(m\overline{CD}) = (m\overline{BC}) \cos(180^o - m\angle C)$ (Use theorem 1).

2. Find $m\ \overline{AC}$ if in ΔABC $m\ \overline{BC} = 6$ cm, $m\ \overline{AB} = 4\sqrt{2}$ cm and $m\angle ABC = 135^o$.

THEOREM 2

8.1(*ii*) In any triangle, the square on the side opposite to acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

Given: ΔABC with an acute angle CAB at A.

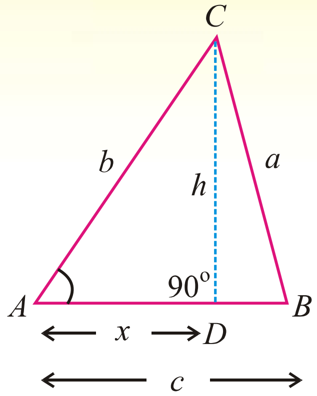
Take $m\overline{BC} = a$, $m\overline{CA} = b$ and $m\overline{AB} = c$

Draw $\overline{CD} \perp \overline{AB}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also, $m\overline{AD} = x$ and $m\overline{CD} = h$

To prove: $(BC)^2 = (AC)^2 + (AB)^2 - 2(AB)(AD)$

i.e., $a^2 = b^2 + c^2 - 2cx$



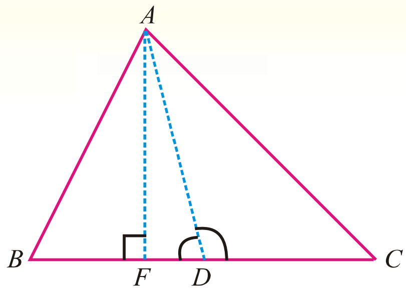
Proof:

Statements		Reasons
In $\angle rt\ \Delta\ CDA$		
$m\angle CDA = 90^o$		Given
$(AC)^2 = (AD)^2 + (CD)^2$		Pythagoras Theorem
i.e., $b^2 = x^2 + h^2$	(i)	
In $\angle rt\ \Delta\ CDB$,		
$m\angle CDB = 90^o$		Given
$(BC)^2 = (BD)^2 + (CD)^2$		Pythagoras Theorem
$a^2 = (c-x)^2 + h^2$		From the figure
or $a^2 = c^2 - 2cx + x^2 + h^2$	(ii)	
$a^2 = c^2 - 2cx + b^2$		Using (i) and (ii)
Hence, $a^2 = b^2 + c^2 - 2cx$		
i.e., $(BC)^2 = (AC)^2 + (AB)^2 - 2(AB)(AD)$		

THEOREM 3

(APOLLONIUS' THEOREM)

8.1.(*iii*) In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Given: In a ΔABC, the median $\overline{AD}$ bisects $\overline{BC}$. i.e., $m\overline{BD} = m\overline{CD}$

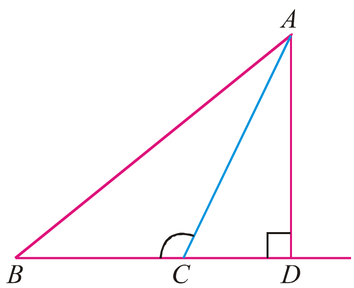
To prove: $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$

Construction: Draw $\overline{AF} \perp \overline{BC}$

Proof:

Statements	Reasons
In ΔADB Since $\angle ADB$ is acute at D	
$\therefore$ $(AB)^2 = (BD)^2 + (AD)^2 - 2\, m\overline{BD}.\, m\overline{FD}$ (i)	Using Theorem 2
Now in ΔADC since $\angle ADC$ is obtuse at D	
$\therefore$ $(AC)^2 = (CD)^2 + (AD)^2 + 2\, m\overline{CD}.\, m\overline{FD}$	Using Theorem 1
$= (BD)^2 + (AD)^2 + 2\, m\overline{BD}\, .\, m\overline{FD}$ (ii)	
Thus $(AB)^2 + (AC)^2 = 2(BD)^2 + 2\,(AD)^2$	Adding (i) and (ii)

Example 1: In ΔABC, $\angle C$ is obtuse, $\overline{AD} \perp \overline{BC}$ produced, whereas $\overline{BD}$ is projection of $\overline{AB}$ on $\overline{BC}$. Prove that $(AC)^2 = (AB)^2 + (BC)^2 - 2m\overline{BC}.m\overline{BD}$



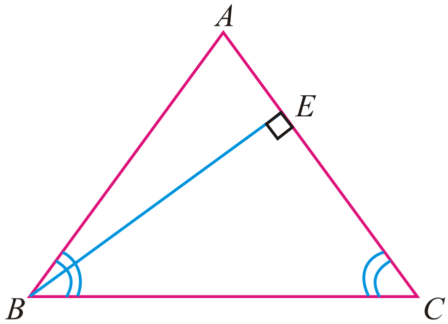
Given: In a ΔABC, $\angle BCA$ is obtuse so that $\angle B$ is acute, $\overline{AD} \perp \overline{BC}$ produced, whereas $\overline{BD}$ is projection of $\overline{AB}$ on $\overline{BC}$ produced.

To prove: $(AC)^2 = (AB)^2 + (BC)^2 - 2m\overline{BC}\, .\, m\overline{BD}$

Proof:

Statements		Reasons
In $\angle rt\ \Delta ABD$		
$(AB)^2 = (AD)^2 + (BD)^2$	(i)	Pythagoras Theorem
In $\angle rt\ \Delta ACD$		
$(AC)^2 = (AD)^2 + (CD)^2$	(ii)	Pythagoras Theorem
or $(AC)^2 = (AD)^2 + (BD - BC)^2$		$m\overline{BC} + m\overline{CD} = m\overline{BD}$
$(AC)^2 = (AD)^2 + (BD)^2 + (BC)^2 - 2BC.BD$	(iii)	
$(AC)^2 = (AB)^2 + (BC)^2 - 2BC.BD$		Using (i) and (iii)

Example 2: In an Isosceles ΔABC, if $m\overline{AB} = m\overline{AC}$ and $\overline{BE} \perp \overline{AC}$, then prove that $(BC)^2 = 2m\overline{AC}.m\overline{CE}$



Given: In an Isosceles ΔABC

$m\overline{AB} = m\overline{AC}$ and $\overline{BE} \perp \overline{AC}$

whereas $\overline{CE}$ is the projection of $\overline{BC}$ upon on $\overline{AC}$.

To prove: $(BC)^2 = 2m\overline{AC}\ .\ m\overline{CE}$

Proof:

Statements	Reasons
In an isosceles ΔABC with $m\overline{AB} = m\overline{AC}$. If $\angle C$ is acute,	
then $(AB)^2 = (AC)^2 + (BC)^2 - 2m\overline{AC}.m\overline{CE}$	By Theorem 2
$(AC)^2 = (AC)^2 + (BC)^2 - 2m\overline{AC}.m\overline{CE}$	Given $m\overline{AB} = m\overline{AC}$
$\Rightarrow$ $(BC)^2 - 2m\overline{AC}.m\overline{CE} = 0$	Cancel $(\overline{AC})^2$ on both sides
or $(BC)^2 = 2m\overline{AC}\ .\ m\overline{CE}$	

EXERCISE 8.2

1. In a ΔABC calculate $m\overline{BC}$ when $m\overline{AB} = 6$cm, $m\overline{AC} = 4$cm and $m\angle A = 60^o$.

2. In a ΔABC, $m\overline{AB} = 6$ cm, $m\overline{BC} = 8$ cm, $m\overline{AC} = 9$ cm and D is the mid point of side $\overline{AC}$. Find length of the median $\overline{BD}$.

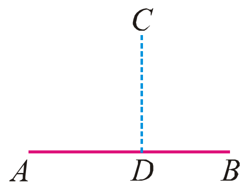
3. In a parallelogram $ABCD$ prove that $(AC)^2 + (BD)^2 = 2\ [(AB)^2 + (BC)^2]$

MISCELLANEOUS EXERCISE 8

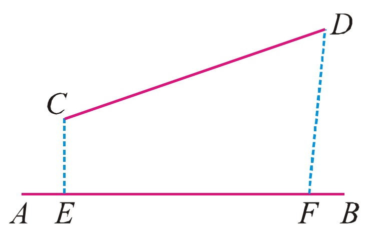
1. In a ΔABC, $m\angle A = 60^o$, prove that $(BC)^2 = (AB)^2 + (AC)^2 - m\overline{AB} \,.\, m\overline{AC}$.
2. In a ΔABC, $m\angle A = 45^o$, prove that $(BC)^2 = (AB)^2 + (AC)^2 - \sqrt{2}\, m\overline{AB} \,.\, m\overline{AC}$.
3. In a ΔABC, calculate $m\overline{BC}$ when $m\overline{AB} = 5$ cm, $m\overline{AC} = 4$ cm, $m\angle A = 60^o$.
4. In a ΔABC, calculate $m\overline{AC}$ when $m\overline{AB} = 5$ cm, $m\overline{BC} = 4\sqrt{2}$ cm, $m\angle B = 45^o$.
5. In a triangle ABC, $m\overline{BC} = 21$ cm, $m\overline{AC} = 17$ cm, $m\overline{AB} = 10$ cm.
   Measure the length of projection of $\overline{AC}$ upon $\overline{BC}$.
6. In a triangle ABC, $m\overline{BC} = 21$ cm. $m\overline{AC} = 17$ cm, $m\overline{AB} = 10$ cm.
   Calculate the projection of $\overline{AB}$ upon $\overline{BC}$.
7. In a ΔABC, $a = 17$ cm, $b = 15$ cm and $c = 8$ cm. Find $m\angle A$.
8. In a ΔABC, $a = 17$ cm, $b = 15$ cm and $c = 8$ cm find $m\angle B$.
9. Whether the triangle with sides 5 cm, 7 cm, 8 cm is acute, obtuce or right angled.
10. Whether the triangle with sides 8 cm, 15 cm, 17 cm is acute, obtuce or right angled.

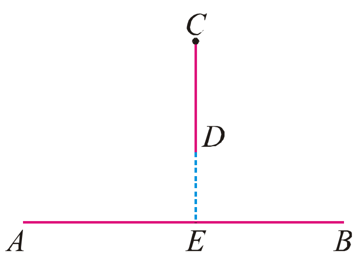
SUMMARY

- The **projection** of a given point on a line segment is the foot of $\perp$ drawn from the point on that line segment. If $\overline{CD} \perp \overline{AB}$, then evidently D is the foot of perpendicular CD from the point C on the line segment AB.



- The projection of a line segment $\overline{CD}$ on a line segment $\overline{AB}$ is the portion $\overline{EF}$ of the latter intercepted between foots of the perpendiculars drawn from C and D. However projection of a vertical line segment $\overline{CD}$ on a line segment $\overline{AB}$ is a point on $\overline{AB}$ which is of **zero dimension**.





- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).