CHAPTER

8

Mathematical Inductions and Binomial Theorem

8.1 Introduction

Francesco Mourolico (1494-1575) devised the method of induction and applied this device first to prove that the sum of the first n odd positive integers equals n^2. He presented many properties of integers and proved some of these properties using the method of *mathematical induction.*

We are aware of the fact that even one exception or case to a mathematical formula is enough to prove it to be false. Such a case or exception which fails the mathematical formula or statement is called a counter example.

The validity of a formula or statement depending on a variable belonging to a certain set is established if it is true for each element of the set under consideration.

For example, we consider the statement $S(n) = n^2 - n + 41$ is a prime number for every natural number n. The values of the expression $n^2 - n + 41$ for some first natural numbers are given in the table as shown below:

n	1	2	3	4	5	6	7	8	9	10	11
S(n)	41	43	47	53	61	71	83	91	113	131	151

From the table, it appears that the statement $S(n)$ has enough chance of being true. If we go on trying for the next natural numbers, we find $n = 41$as a counter example which fails the claim of the above statement. So we conclude that to derive a general formula without proof from some special cases is not a wise step. This example was discovered by Euler (1707-1783).

Now we consider another example and try to formulate the result. Our task is to find the sum of the first n odd natural numbers. We write first few sums to see the pattern of sums.

n (The number of terms)	Sum
1	$1 = 1^2$
2	$1+3 = 4 = 2^2$
3	$1+3+5 = 9 = 3^2$
4	$1+3+5+7 = 16 = 4^2$
5	$1+3+5+7+9 = 25 = 5^2$
6	$1+3+5+7+9+11 = 36 = 6^2$

The sequence of sums is $(1)^2$, $(2)^2$, $(3)^2$, $(4)^2$, ...

We see that each sum is the square of the number of terms in the sum. So the following statement seems to be true.

For each natural number n,

$1 + 3 + 5 + \dots + (2n - 1) = n^2$ . . . . (i) $\quad$ ($\because$ nth term$=1+(n-1)2$)

But it is not possible to verify the statement (i) for each positive integer n, because it involves infinitely many calculations which never end.

The method of mathematical induction is used to avoid such situations.Usually it is used to prove the statements or formulae relating to the set {1,2,3,...} but in some cases, it is also used to prove the statements relating to the set (0 ,1,2 ,3,...}.

8.2 Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer n is such that

1) $S(1)$ is true i.e., $S(n)$ is true for $n = 1$ and
2) $S(k + 1)$ is true whenever $S(k)$ is true for any positive integer k, then $S(n)$ is true for all positive integers.

Procedure:

1. Substituting $n = 1$, show that the statement is true for $n = 1$.
2. Assuming that the statement is true for any positive integer k, then show that it is true for the next higher integer. For the second condition, one of the following two methods can be used:

M_1 Starting with one side of $S(k+1)$, its other side is derived by using $S(k)$.

M_2 $S(k + 1)$ is established by performing algebraic operations on $S(k)$.

Example 1: Use mathematical induction to prove that $3+6+9+\dots+3n=\frac{3n(n+1)}{2}$ for every positive integer n.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n): \quad 3+6+9....+3n=\frac{3n(n+1)}{2} \qquad \text{(i)}$$

1. When n = 1, S(1) becomes

$$S(1): 3=\frac{3(1)(1+1)}{2}=3$$

Thus S(1) is true i.e., condition (1) is satisfied.

2. Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$3+6+9....+3k=\frac{3k(k+1)}{2} \qquad \text{(A)}$$

The statement for $n = k+1$ becomes

$$3+6+9....+3k+3(k+1)=\frac{3k(k+1)\left[(k+1)+1\right]}{2}$$
$$=\frac{3(k+1)(k+2)}{2} \qquad \text{(B)}$$

Adding $3(k+1)$ on both the sides of (A) gives

$$3+6+9+....+3k+3(k+1)=\frac{3k(k+1)}{2}+3(k+1)$$
$$=3(k+1)(\frac{k}{2}+1)$$
$$=\frac{3(k+1)(k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so the condition (2) is satisfied.

Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

Example 2: Use mathematical induction to prove that for any positive integer n,

$$1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

Solution: Let $S(n)$ be the given statement,

$$S(n): \quad 1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

1. If n = 1, S (1) becomes

$$S(1):(1)^2=\frac{1(1+1)(2\times1+1)}{6}=\frac{1\times2\times3}{6}=1$$

Thus S(1) is true, i.e., condition (1) is satisfied.

2. Let us assume that $S(k)$ is true for any $k \in N$, that is,

$$1^2+2^2+3^2+....+k^2=\frac{k(k+1)(2k+1)}{2} \qquad \text{(A)}$$

$$S(k+1): \; 1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+1)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6} \qquad \text{(B)}$$

Adding $(k+1)^2$ to both the sides of equation (A), we have

$$1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$
$$=\frac{(k+1)(2k^2+k+6k+6)}{6}$$
$$=\frac{(k+1)(2k^2+7k+6)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

Thus the condition (2) is satisfied. Since both the conditions are satisfied, therefore, by mathematical induction, the given statement holds for all positive integers.

Example 3: Show that $\frac{n^3+2n}{3}$ represents an integer $\forall n \in N$.

Solution: Let $S(n)=\frac{n^3+2n}{3}$

1. When n = 1, S(1) becomes

$$S(1)=\frac{1^3+2(1)}{3}=\frac{3}{3}=1\in Z$$

2. Let us assume that $S(n)$ is ture for any $n = k \in W$, that is,

$$S(k)=\frac{k^3+2k}{3} \text{ represents an integer.}$$

Now we want to show that $S(k+1)$ is also an integer. For $n=k+1$, the statement becomes

$$S(k+1)=\frac{(k+1)^3+2(k+1)}{3}$$
$$=\frac{k^3+3k^2+3k+1+2k+2}{3}=\frac{(k^3+2k)+(3k^2+3k+3)}{3}$$
$$=\frac{(k^3+2k)+3(k^2+k+1)}{3}$$
$$=\frac{k^3+2k}{3}+(k^2+k+1)$$

As $\frac{k^3+2k}{3}$ is an integer by assumption and we know that $(k^2 + k + 1)$is an integer as $k \in W$.

$S(k + 1)$ being sum of integers is an integer, thus the condition (2) is satisfied.

Since both the conditions are satisfied, therefore, we conclude by mathematical induction that $\frac{n^3+2n}{3}$ represents an integer for all positive integral values of n.

Example 4: Use mathematical induction to prove that

$$3+3.5+3.5^2+....+3.5^n=\frac{3(5^{n+1}-1)}{4} \text{ whenever } n \text{ is non-negative integer.}$$

Solution: Let $S(n)$ be the given statement, that is,

$$S(n):\quad 3+3.5+3.5^2+....+3.5^n=\frac{3(5^{n+1}-1)}{4}$$

The dot (.)between two number, stands, for multipication symbol.

1. For n= 0, S(0) becomes $S(0): 3.5^0=\frac{3(5^{0+1}-1)}{4}$ or $3=\frac{3(5-1)}{4}=3$

Thus S(0) is true i.e., conditions (1) is satisfied.

2. Let us assume that $S(k)$ is true for any $k \in W$, that is,

$$S(k): 3+3.5+3.5^2+....+3.5^k=\frac{3(5^{k+1}-1)}{4} \quad \text{(A)}$$

Here $S(k+1)$ becomes

$$S(k+1): 3+3.5+3.5^2+....+3.5^k+3.5^{k+1}=\frac{3(5^{(k+1)+1}-1)}{4}$$
$$=\frac{3(5^{k+2}-1)}{4} \quad \text{(B)}$$

Adding 3.5^{k+1} on both sides of (A), we get

$$3+3.5+3.5^2+....+3.5^k+3.5^{k+1}=\frac{3(5^{k+1}-1)}{4}+3.5^{k+1}$$
$$=\frac{3(5^{k+1}-1+4.5^{k+1})}{4}$$
$$=\frac{3[5^{k+1}(1+4)-1]}{4}$$
$$=\frac{3(5^{k+2}-1)}{4}$$

This shows that $S(k + 1)$ is true when $S(k)$ is true. Since both the conditions are satisfied, therefore, by the principle of mathematical induction, $S(n)$ in true for each $n \in W$.

Care should be taken while applying this method. Both the conditions (1) and (2) of the principle of mathematical induction are essential. The condition (1) gives us a starting point but the condition (2) enables us to proceed from one positive integer to the next. In the condition (2) we do not prove that $S(k+1)$ is true but prove only that if $S(k)$is true, then $S(k+1)$ is true. We can say that any proposition or statement for which only one condition is satisfied, will not be truefor all n belonging to the set of positive integers.

For example, we consider the statement that 3^n is an even integer for any positive integer n. Let $S(n)$ be the given statement.

Assume that $S(k)$ is true, that is, 3^k in an even integer for $n = k$. When 3^k is even, then $3^k + 3^k + 3^k$ is even which implies that $3^k .3= 3^{k+1}$ is even.

This shows that $S(k+1)$ will be true when $S(k)$ is true. But 3^1 is not an even integer which reflects that the first condition does not hold. Thus our supposition is false.

Note:- There is no integer n for which 3^n is even.

Sometimes, we wish to prove formulae or statements which are true for all integers n greater than or equal to some integer i, where $i \neq 1$. In such cases, $S(1)$ is replaced by $S(i)$ and the condition (2) remains the same. To tackle such situations, we use the principle of extended mathematical induction which is stated as below:

8.3 Principle of Extended Mathematical Induction

Let i be an integer. If a formula or statement $S(n)$ for $n \geq i$ is such that

1) $S(i)$ is true and
2) $S(k+1)$ is true whenever $S(k)$ is true for any integer $n \geq i$.

Then $S(n)$ is true for all integers $n \geq i$.

Example 5: Show that $1 + 3 + 5 + \ldots (2n+5) = (n+3)^2$ for integral values of $n \geq -2$.

Solution:

1. Let $S(n)$ be the given statement, then for $n = -2$, $S(-2)$ becomes, $2(-2)+5=(-2+3)^2$, i.e., $1 = (1)^2$ which is true.
   Thus $S(-2)$ is true i.e., the condition (1) is satisfied
2. Let the equation be true for any $n = k \in Z$, $k \geq -2$, so that
   $1+3+5+\ldots+(2k+5)= (k+3)^2$ (A)

$$S(k+1): 1+3+5+\ldots+(2k+5)+(2\overline{k+1}+5)=(\overline{k+1}+3)^2=(k+4)^2 \quad \text{(B)}$$

Adding $(2\overline{k+1}+5)=(2k+7)$ on both sides of equation (A) we get,

$$\begin{aligned} 1+3+5+\ldots+(2k+5)+(2k+7) &= (k+3)^2+(2k+7) \\ &= k^2+6k+9+2k+7 \\ &= k^2+8k+16 \\ &= (k+4)^2 \end{aligned}$$

Thus the condition (2) is satisfied. As both the conditions are satisfied, so we conclude that the equation is true for all integers $n \geq -2$.

Example 6: Show that the inequality $4^n > 3^n + 4$ is true, for integral values of $n \geq 2$.

Solution: Let $S(n)$ represents the given statement i.e., $S(n)$: $4^n > 3^n + 4$ for integral values of $n \geq 2$.

1. For $n = 2$, $S(2)$ becomes
   $S(2)$: $4^2 > 3^2 + 4$, i.e., $16 > 13$ which is true.
   Thus $S(2)$ is true, i.e., the first condition is satisfied.
2. Let the statement be true for any $n = k(\geq 2) \in Z$, that is
   $4^k > 3^k + 4$ (A)
   Multiplying both sides of inequality (A) by 4, we get
   or $4.4^k > 4(3^k + 4)$
   or $4^{k+1} > (3+1)3^k + 16$
   or $4^{k+1} > 3^{k+1} + 4 + 3^k + 12$
   or $4^{k+1} > 3^{k+1} + 4$ $(\because 3^k + 12 > 0)$ (B)
   The inequality (B), satisfies the condition (2).

Since both the conditions are satisfied, therefore, by the principle of extended mathematical induction, the given inequality is true for all integers $n \geq 2$.

Exercise 8.1

Use mathematical induction to prove the following formulae for every positive integer n.

1. $1+5+9+\ldots+(4n-3)=n(2n-1)$

2. $1+3+5+....+(2n-1)=n^2$

3. $1+4+7+....+(3n-2)=\dfrac{n(3n-1)}{2}$

4. $1+2+4+....+2^{n-1}=2^n-1$

5. $1+\dfrac{1}{2}+\dfrac{1}{4}+....+\dfrac{1}{2^{n-1}}=2\left[1-\dfrac{1}{2^n}\right]$

6. $2+4+6+....+2n=n(n+1)$

7. $2+6+18+....+2\times 3^{n-1}=3^n-1$

8. $1\times 3+2\times 5+3\times 7+....+n\times(2n+1)=\dfrac{n(n+1)(4n+5)}{6}$

9. $1\times 2+2\times 3+3\times 4+....+n\times(n+1)=\dfrac{n(n+1)(n+2)}{3}$

10. $1\times 2+3\times 4+5\times 6+....+(2n-1)\times 2n=\dfrac{n(n+1)(4n-1)}{3}$

11. $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+....+\dfrac{1}{n(n+1)}=1-\dfrac{1}{n+1}$

12. $\dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+\dfrac{1}{5\times 7}+....+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$

13. $\dfrac{1}{2\times 5}+\dfrac{1}{5\times 8}+\dfrac{1}{8\times 11}+....+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{2(3n+2)}$

14. $r+r^2+r^3+....+r^n=\dfrac{r(1-r^n)}{1-r},\qquad (r\neq 1)$

15. $a+(a+d)+(a+2d)+....+[a+(n-1)d]=\dfrac{n}{2}[2a+(n-1)d]$

16. $1.\underline{|1}+2\underline{|2}+3\underline{|3}+....+n\underline{|n}=\underline{|n+1}-1$

17. $a_n=a_1+(n-1)d$ when, $a_1, a_1+d, a_1+2d,$ .... form an A.P.

18. $a_n=a_1r^{n-1}$ when $a_1, a_1r, a_1r^2,$... form a G.P.

19. $1^2+3^2+5^2+....+(2n-1)^2=\dfrac{n(4n^2-1)}{3}$

20. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+....+\binom{n+2}{3}=\binom{n+3}{4}$

21. Prove by mathematical induction that for all positive integral values of n

i) n^2+n is divisible by 2. ii) $5^n - 2''$ is divisible by 3.

iii) 5^n-1 is divisible by 4. iv) $8 \times 10^n - 2$ is divisible by 6.

v) $n^3 - n$ is divisible by 6.

22. $\dfrac{1}{3}+\dfrac{1}{3^2}+....+\dfrac{1}{3^n}=\dfrac{1}{2}\left[1-\dfrac{1}{3^n}\right]$

23. $1^2-2^2+3^2-4^2+....+(-1)^{n-1}\ .n^2=\dfrac{(-1)^{n-1}.n(n+1)}{2}$

24. $1^3+3^3+5^3+....+(2n-1)^3=n^2[2n^2-1]$

25. $x+1$ is a factor of $x^{2n}-1; (x\neq -1)$

26. $x - y$ is a factor of $x^n - y^n$; $(x \neq y)$

27. $x + y$ is a factor of $x^{2n-1} + y^{2n-1}$ $(x \neq -y)$

28. Use mathematical induction to show that
$1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all non-negative integers n.

29. If A and B are square matrices and $AB = BA$, then show by mathematical induction that $AB^n = B^nA$ for any positive integer n.

30. Prove by the Principle of mathematical induction that $n^2 - 1$ is divisible by 8 when n is an odd positive integer.

31. Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any integer $n \geq 0$ if x is a positive number. Use the principle of extended mathematical induction to prove that:

32. $n! > 2^n - 1$ for integral values of $n \geq 4$.

33. $n^2 > n + 3$ for integral values of $n \geq 3$.

34. $4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

35. $3^n < n!$ for integral values of $n > 6$.

36. $n! > n^2$ for integral values of $n \geq 4$.

37. $3 + 5 + 7 + ... + (2n + 5) = (n + 2)(n + 4)$ for integral values of $n \geq -1$.

38. $1 + nx \leq (1 + x)^n$ for $n \geq 2$ and $x > -1$

8.4 Binomial Theorem

An algebraic expression consisting of two terms such as $a + x$, $x - 2y$, $ax + b$ etc., is called a binomial or a binomial expression.

We know by actual multiplication that

$(a + x)^2 = a^2 + 2ax + x^2$ (i)

$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$ (ii)

The right sides of (i) and (ii) are called binomial expansions of the binomial $a + x$ for the indices 2 and 3 respectively.

In general, the rule or formula for expansion of a binomial raised to any positive integral power n is called the binomial theorem for positive integral index n.For any positive integer n,

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + ... + \binom{n}{r-1}a^{n-(r-1)}x^{r-1} \qquad \text{(A)}$$

$$+\binom{n}{r}a^{n-r}x^r + ... + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n$$

or briefly

$$(a+x)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}x^r$$

where a and x are real numbers.

The rule of expansion given above is called the binomial theorem and it also holds if a or x is complex.

Now we prove the Binomial theorem for any positive integer n, using the principle of mathematical induction.

Proof: Let $S(n)$ be the statement given above as (A).

1. If $n = 1$, we obtain

$$S(1): (a+x)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}x = a + x$$

Thus condition (1) is satisfied.

2. Let us assume that the statement is true for any $n = k \in N$, then

$$(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + ... + \binom{k}{r-1}a^{k-(r-1)}x^{r-1} + \binom{k}{r}a^{k-r}x^r$$

$$+ .... + \binom{k}{k}ax^k \ + \binom{k}{k}x^k \qquad \text{(B)}$$

$$S(k+1): (a+k)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k \times x + \binom{k+1}{2}a^{k-1} \times x^2 + ... +$$

$$\binom{k+1}{r-1}a^{k-r+2} \times x^{r-1} + \binom{k+1}{r}a^{k-r+1} \times x^r + ... + \binom{k+1}{k}a \times x^k + \binom{k+1}{k+1}x^{k+1} \qquad \text{(C)}$$

Multiplying both sides of equation (B) by $(a+x)$, we have

$$(a+x)(a+x)^k = (a+x)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + ... + \binom{k}{r-1}a^{k-r+1}x^{r-1}\right.$$

$$\left. + \binom{k}{r}a^{k-r} \ x^r + ... + \binom{k}{k-1}ax^{k-1} + \binom{k}{k}x^k\right]$$

$$= \left[\binom{k}{0}a^{k+1} + \binom{k}{1}a^k x + \binom{k}{2}a^{k-1}x^2 + ... + \binom{k}{r-1}a^{k-r+2}x^{r-1}\right.$$

$$\left. + \binom{k}{r}a^{k-r+1}x^r + ... + \binom{k}{k-1}a^2x^{k-1} + \binom{k}{k}ax^k\right]$$

$$+\left[\binom{k}{0}a^k x+\binom{k}{1}a^{k-1}x^2+\binom{k}{2}a^{k-2}\times x^3+\dots+\binom{k}{r-1}a^{k-r+1}\times x^r\right.$$

$$\left.+\binom{k}{r}a^{k-r}x^{r+1}+\dots+\binom{k}{k-1}ax^k+\binom{k}{k}x^{k+1}\right]$$

$$=\binom{k}{0}a^{k+1}+\left[\binom{k}{1}+\binom{k}{0}\right]a^k x+\left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}x^2+\dots$$

$$+\left[\binom{k}{r}+\binom{k}{r-1}\right]a^{k-r+1}x^r+\dots+\left[\binom{k}{k}+\binom{k}{k-1}\right]ax^k+\binom{k}{k}x^{k+1}$$

As $\binom{k}{0}=\binom{k+1}{0},\binom{k}{k}=\binom{k+1}{k+1}\text{and}\binom{k}{r}+\binom{k}{r-1}=\binom{k+1}{r}\text{for}1\le r\le k$

$$\therefore (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^k x+\binom{k+1}{2}a^{k-1}x^2+\dots$$

$$+\binom{k+1}{r}a^{k-r+1}x^r+\dots+\binom{k+1}{k}a\;x^k+\binom{k+1}{k+1}x^{k+1}$$

We find that if the statement is true of $n = k$, then it is also true for $n = k + 1$. Hence we conclude that the statement is true for all positive integral values of n.

Note: $\binom{n}{0},\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n}$ are called the binomial coefficients.

The following points can be observed in the expansion of $(a + x)^n$

1. **The number of terms in the expansion is one greater than its index.**
2. **The sum of exponents of a and x in each term of the expansion is equal to its index.**
3. **The exponent of a decreases from index to zero.**
4. **The exponent of x increases from zero to index.**
5. **The coefficients of the terms equidistant from beginning and end of the expansion are equal as** $\binom{n}{r}=\binom{n}{n-r}$
6. **The $(r + 1)$th term in the expansion** $\binom{n}{r}a^{n-r}x^r$ **and we denote it as** T_{r+1} **i.e.,**

$$T \quad \left(\quad\right)a^{n\ r}x^r$$

As all the terms of the expansion can be got from it by putting $r = 0, 1,2.....$ n, so we call it as the **general term** of the expansion.

Example 1: Expand $\left(\frac{a}{2}-\frac{2}{a}\right)^6$ also find its general term.

Solution: $\left(\frac{a}{2}-\frac{2}{a}\right)^6=\left(\frac{a}{2}+\left(\frac{-2}{a}\right)\right)^6$

$$=\left(\frac{a}{2}\right)^6+\binom{6}{1}\left(\frac{a}{2}\right)^5\left(-\frac{2}{a}\right)+\binom{6}{2}\left(\frac{a}{2}\right)^4+\left(-\frac{2}{a}\right)^2+\binom{6}{3}\left(\frac{a}{2}\right)^3\left(-\frac{2}{a}\right)^3$$

$$+\binom{6}{4}\left(\frac{a}{2}\right)^2\left(-\frac{2}{a}\right)^4+\binom{6}{5}\left(\frac{a}{2}\right)\left(-\frac{2}{a}\right)^5+\left(-\frac{2}{a}\right)^6$$

$$=\frac{a^6}{64}+6\left(\frac{a^5}{32}\right)\left(-\frac{2}{a}\right)+\frac{6.5}{2.1}.\frac{a^4}{16}.\frac{4}{a^2}.+\frac{6.5.4}{3.2.1}.\frac{a^3}{8}.\left(-\frac{8}{a^3}\right)+\frac{6.5}{2.1}.\frac{a^2}{4}.\frac{16}{a^4}$$

$$+6.\frac{a}{2}\left(\frac{-32}{a^5}\right)+\frac{64}{a^6}$$

$$=\frac{a^6}{64}-\frac{3}{8}a^4+\frac{15}{4}a^2-20+\frac{60}{a^2}-\frac{96}{a^4}+\frac{64}{a^6}$$

T_{r+1}, the general term is given by

$$T_{r+1}=\binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\left(-\frac{2}{a}\right)^r=\binom{6}{r}\frac{a^{6-r}}{2^{6-r}}(-1)^r\frac{2^r}{a^r}$$

$$=(-1)^r\binom{6}{r}\frac{a^{6-r}.a^{-r}}{2^{6-r}.2^{-r}}=(-1)^r.\binom{6}{r}\frac{a^{6-2r}}{2^{6-2r}}=(-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-2r}$$

Example 2: Evaluate $(9.9)^5$

Solution: $(9.9)^5 = (10–.1)^5$
= $(10)^5$ + 5 x (10)4 x (–. 1) + $10(10)^3$ x $(–. 1)^2$ + 10(10)2 x $(–. 1)^3$ + $5(10)(–.1)^4$ + $(–.1)^5$
= 100000 – (.5) (10000) + (10000X.01) + 1000(–.001) + 50 (.0001) –.00001
= 100000 – 5000 +100 –1 + .005 – .00001
= 100100.005 – 5001.00001
= 95099.00499

Example 3: Find the specified term in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$;

i) the term involving x^5 ii) the fifth term
iii) the sixth term from the end. iv) coefficient of term involving x^{-1}

Solution:

i) Let T_{r+1} be the term involving x^5 in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$, then

$$T_{r+1}=\binom{11}{r}\left(\frac{3}{2}x\right)^{11-r}\left(-\frac{1}{3x}\right)^r=\binom{11}{r}\frac{3^{11-r}}{2^{11-r}}x^{11-r}.(-1)^r.3^{-r}.x^{-r}$$

$$=(-1)^r.\binom{11}{r}\frac{3^{11-2r}}{2^{11-r}}.x^{11-2r}$$

As this term involves x^5 , so the exponent of x is 5, that is,

$$11-2r=5$$

or $-2r=5-11\Rightarrow r=3$

Thus T_4 involves x^5

$$\therefore\ T_4=(-1)^3.\binom{11}{3}\frac{3^{11-6}}{2^{11-3}}.x^{11-6}\qquad=(-1).\frac{11.10.9}{3.2.1}.\frac{3^5}{2^8}.x^5$$

$$=-\frac{165\times243}{256}x^5\qquad=-\frac{40095}{256}x^5$$

ii) Putting **r** = 4 in T_{r+1}, we get T_5

$$\therefore\quad T_5=(-1)^4\binom{11}{4}\frac{3^{11-8}}{2^{11-4}}.x^{11-8}\quad=\frac{11.10.9.8}{4.3.2.1}.\frac{3^3}{2^7}.x^3$$

$$=\frac{11\times10\times3}{1}.\frac{27}{128}x^3=\frac{165\times27}{64}x^3$$

$$=\frac{4455}{64}x^3$$

iii) The 6th term from the end term will have (11 + 1) – 6 i.e., 6 terms before it,
∴ It will be (6 + 1) th term i.e.. the 7th term of the expansion.

Thus $T_7=(-1)^6\binom{11}{6}\frac{3^{11-12}}{2^{11-6}}x^{11-12}=\frac{11.10.9.8.7}{5.4.3.2.1}\frac{3^{-1}}{2^5}.x^{-1}$

$$=\frac{11\times6\times7}{1}.\frac{1}{3\times32}.\frac{1}{x}=\frac{77}{16x}$$

iv) $\frac{77}{16}$ is the coefficient of the term involving x^{-1}

8.3.1 The Middle Term in the Expansion of $(a + x)^n$

In the expansion of $(a + x)^n$, the total number of terms is $n + 1$

Case I: (n is even)

If n is even then $n+1$ is odd, so $\left(\frac{n+1}{2}\right)$th term will be the only one middle term in the expansion.

Case II: (n is odd)

If n is odd then $n + 1$ is even so $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms of the expansion will be the two middle terms.

Example 4: Find the following in the expansion of $\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$;

i) the term independent of x. ii) the middle term

Solution: i) Let T_{r+1} be the term independent of x in the expansion of

$\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$, then

$$T_{r+1}=\binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\left(\frac{2}{x^2}\right)^{r}=\binom{12}{r}\frac{x^{12-r}}{2^{12-r}}.2^r.x^{-2r}$$

$$=\binom{12}{r}2^{2r-12}.x^{12-3r}$$

As the term is independent of x, so exponent of x, will be zero.

That is, $12 - 3r = 0 \Rightarrow r = 4$.

Therefore the required term $T_5=\binom{12}{4}2^{8-12}.x^{12-12}$

$$=\frac{12\times11\times10\times9}{4\times3\times2\times1}.2^{-4}.x^0$$

$$=\frac{11\times45}{2^4}=\frac{495}{16}$$

ii) In this case, $n = 12$ which is even, so

$\therefore$ $\left(\frac{12}{2}+1\right)$th term is the middle term in the expansion,

i.e., T_7 is the required term.

$$T_7=\binom{12}{6}\left(\frac{x}{2}\right)^{12-6}.\left(\frac{2}{x^2}\right)^{6}$$

$$=\binom{12}{6}\frac{x^6}{2^6}.\frac{2^6}{x^{12}}=\frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1}.x^{6-12}$$

$$=\frac{12\times11\times7}{x^6}=\frac{924}{x^6}$$

8.3.2 Some Deductions from the binomial expansion of $(a + x)^n$.

We know that

$$(a+x)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\dots$$

$$+\binom{n}{r}a^{n-r}x^r+\dots+\binom{n}{n-1}ax^{n-1}+\binom{n}{n}x^n \qquad \text{(I)}$$

(i) If we put a = 1, in (I), then we have;

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+....+\binom{n}{r}x^r+....+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n \quad \text{(II)}$$

$$=1+nx+\frac{n(n-1)}{2!}x^2+....+\frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r+....+nx^{n-1}+x^n$$

$$\left(\because \binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{n(n-1)....(n-r+1)(n-r)!}{r!(n-r)!}=\frac{n(n-1)....(n-r+1)}{r!}\right)$$

ii) Putting a = 1 and replacing x by $-x$, in (I), we get.

$$(1-x)^n=\binom{n}{0}+\binom{n}{1}(-x)+\binom{n}{2}(-x)^2+\binom{n}{3}(-x)^3+....+\binom{n}{n-1}(-x)^{n-1}+\binom{n}{n}(-x)^n$$

$$=\binom{n}{0}-\binom{n}{1}x+\binom{n}{2}x^2-\binom{n}{3}x^3+....+(-1)^{n-1}\binom{n}{n-1}x^{n-1}+(-1)^n\binom{n}{n}x^n.... \quad \text{(III)}$$

iii) We can find the sum of the binomial cofficients by putting a = 1and x = 1 in (I).

i.e., $(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+....+\binom{n}{n-1}+\binom{n}{n}$

or $2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+....+\binom{n}{n-1}+\binom{n}{n}$

Thus the sum of coefficients in the binomial expansion equals to 2^n .

iv) Putting a = 1 and $x = -1$, in (i) we have

$$(1-1)^n=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+....+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n}$$

Thus $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+....+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n}=0$

If n is odd positive integer, then

$$\binom{n}{0}+\binom{n}{2}+....+\binom{n}{n-1}=\binom{n}{1}+\binom{n}{3}+....+\binom{n}{n}$$

If n is even positive integer, then

$$\binom{n}{0}+\binom{n}{2}+....+\binom{n}{n}=\binom{n}{1}+\binom{n}{3}+....+\binom{n}{n-1}$$

Thus sum of odd coefficients of a binomial expansion equals to the sum of its even coefficients.

Example 5: Show that: $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+....+n\binom{n}{n}=n.2^{n-1}$

Solution:

$$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+....+n\binom{n}{n}=n+2\frac{n(n-1)}{2!}+3\frac{n(n-1)(n-2)}{3!}+....+n.1$$

$$=n.\left[1+(n-1)+\frac{(n-1)(n-2)}{2!}+....+1\right]$$

$$=n.\left[\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+....+\binom{n-1}{n-1}\right]$$

$$=n.2^{n-1}$$

Exercise 8.2

1. Using binomial theorem, expand the following:

i) $(a+2b)^5$ ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$ iii) $\left(3a-\frac{x}{3a}\right)^4$

iv) $\left(2a-\frac{x}{a}\right)^7$ v) $\left(\frac{x}{2y}+\frac{2y}{x}\right)^8$ vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

2. Calculate the following by means of binomial theorem:
i) $(0.97)^3$ ii) $(2.02)^4$ iii) (9.98) iv) $(21)^5$

3. Expand and simplify the following:

i) $\left(a+\sqrt{2}x\right)^4+\left(a-\sqrt{2}x\right)^4$ ii) $\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5$

iii) $(2+i)^5-(2-i)^5$ iv) $\left(x+\sqrt{x^2-1}\right)^3+\left(x-\sqrt{x^2-1}\right)^3$

4. Expand the following in ascending power of x:
i) $(2+x-x^2)^4$ ii) $(1-x+x^2)^4$ iii) $(1-x-x^2)^4$

5. Expand the following in descending powers of x:

i) $(x^2+x-1)^3$ ii) $\left(x-1-\frac{1}{x}\right)^3$

6. Find the term involving:
i) x^4 in the expansion of $(3-2x)^7$

ii) x^{-2} in the expansion of $\left(x-\frac{2}{x^2}\right)^{13}$

iii) a^4 in the expansion of $\left(\frac{2}{x}-a\right)^9$

iv) y^3 in the expansion of $\left(x-\sqrt{y}\right)^{11}$

7. Find the coefficient of;

i) x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

ii) x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

8. Find 6th term in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

9. Find the term independent of x in the following expansions.

i) $\left(x-\frac{2}{x}\right)^{10}$ ii) $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$ iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4$

10. Determine the middle term in the following expansions:

i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$ ii) $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$ iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

11. Find $(2n+1)$ th term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$

12. Show that the middle term of $(1+x)^{2n}$ is $=\frac{1.3.5....(2n-1)}{n!}2^n x^n$

13. Show that: $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+....+\binom{n}{n-1}=2^{n-1}$

14. Show that: $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+....+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

8.4 The Binomial Theorem when the index *n* is a negative integer or a fraction.

When *n* is a negative integer or a fraction, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$+\frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r + ...$$

provided $|x|<1$.

The series of the type

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ ...$$

is called the binomial series.

Note (1): The proof of this theorem is beyond the scope of this book.

(2) Symbols $\binom{n}{0},\binom{n}{1},\binom{n}{2}$ etc are meaningless when n is a negative integer or a fraction.

(3) The general term in the expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r$$

Example 1: Find the general term in the expansion of $(1+x)^{-3}$ when $|x|<1$

Solution: $T_{r+1} = \frac{(-3)(-4)(-5)....(-3-r+1)}{r!}x^r$

$$= \frac{(-1)^r.3.4.5....(r+2)}{r!}x^r$$

$$= (-1)^r.\frac{1.2.3.4.5....(r+2)}{1.2.r!}x^r$$

$$= (-1)^r\frac{r!.(r+1)(r+2)}{2.r!}x^r$$

$$= (-1)^r.\frac{(r+1)(r+2)}{2}x^r$$

Some particular cases of the expansion of $(1+x)^n$, n<0.

i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$

ii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r(r+1)x^r + ...$

iii) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + (-1)^r\frac{(r+1)(r+2)}{2}x^r + ...$

iv) $(1-x)^{-1} = 1 + x + x^2 - x^3 + ... + x^r + ...$

v) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r+1)x^r + ...$

vi) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \frac{(r+1)(r+2)}{2}x^r + ...$

8.5 Application of the Binomial Theorem

Approximations: We have seen in the particular cases of the expansion of $(1+x)^n$ that the power of x go on increasing in each expansion. Since $|x| < 1$, so

$$|x|^r < |x| \text{ for } r = 2,3,4...$$

This fact shows that terms in each expansion go on decreasing numerically if $|x| < 1$. Thus some initial terms of the binomial series are enough for determining the approximate values of binomial expansions having indices as negative integers or fractions.

Summation of infinite series: The binomial series are conveniently used for summation of infinite series..The series (*whose sum is required*) is compared with

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

to find out the values of *n* and x. Then the sum is calculated by putting the values of *n* and x in $(1+x)^n$.

Example 2: Expand $(1-2x)^{1/3}$ to four terms and apply it to evaluate $(.8)^{1/3}$ correct to three places of decimal.

Solution: This expansion is valid only if $|2x|<1 \text{ or } 2|x|<1 \text{ or } |x|<-$, that is

$$(1-2x)^{1/3}=1+\frac{1}{3}(-2x)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-2x)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-2x)^3-\ldots\ldots$$

$$=1-\frac{2}{3}x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2.1}(4x^2)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3.2.1}(-8x^3)-\ldots.$$

$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{1.2.5}{3.3.3}.\frac{1}{3.2.1}(8x^3)-\ldots.$$

$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{40}{81}x^3-\ldots.$$

Putting $x=.1$ in the above expansion we have

$$\left(1-2(.1)\right)^{1/3}=1-\frac{2}{3}(.1)-\frac{4}{9}(.1)^2-\frac{40}{81}(.1)^3-\ldots$$

$$=1-\frac{.2}{3}-\frac{.04}{9}-\frac{.04}{81}\ldots \qquad (\because 40\times.001=.04)$$

$$\approx 1-.06666-.00444-.00049=1-.07159=.92841$$

Thus $(.8)^{1/3}\approx .928$

Alternative method:

$$(.8)^{1/3}=(1-.2)^{1/3}=1-\frac{.2}{3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-.2)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-.2)^3+\ldots.$$

Simplify onward by yourself.

Example 3: Expand $(8-5x)^{-2/3}$ to four terms.

Solution: $(8-5x)^{-2/3}=\left(8\left(1-\frac{5x}{8}\right)\right)^{-\frac{2}{3}}=8^{-\frac{2}{3}}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}=(8^{\frac{1}{3}})^{-2}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}$

$$=\frac{1}{4}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}$$

$$=\frac{1}{4}\left[1+\left(-\frac{2}{3}\right)\left(-\frac{5}{8}x\right)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2!}\left(-\frac{5}{8}x\right)^2+\frac{\left(-\frac{2}{3}\right)\left(\frac{-5}{3}\right)\left(-\frac{8}{3}\right)}{3!}\left(-\frac{5}{8}x\right)^3+\ \ldots\right]$$

$$=\frac{1}{4}\left[1+\frac{5}{12}x+\frac{5}{9}\times\frac{25}{64}x^2+\frac{40}{81}\times\frac{125}{8\times 64}x^3+\ldots\right]$$

$$=\frac{1}{4}+\frac{5}{48}x+\frac{125}{2304}x^2+\frac{625}{20736}x^3+\ldots.$$

The expansion of $\left(1-\frac{5}{8}x\right)^{-2/3}$ is valid when $\left|\frac{5}{8}x\right|<1$

or $\frac{5}{8}|x|<1 \Rightarrow |x|<\frac{8}{5}$

Example 4: Evaluate $\sqrt[3]{30}$ correct to three places of decimal.

Solution: $\sqrt[3]{30}=(30)^{1/3}=(27+3)^{\frac{1}{3}}$

$$=\left[27\left(1+\frac{3}{27}\right)\right]^{1/3}=(27)^{1/3}\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left[1+\frac{1}{3}\cdot\frac{1}{9}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{9}\right)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{9}\right)^3+\ldots\right]$$

$$=3\left[1+\frac{1}{3}\cdot\frac{1}{9}-\frac{1}{9}\left(\frac{1}{9}\right)^2+\frac{5}{81}\left(\frac{1}{9}\right)^3+\ldots\right]=3\left[1+\frac{1}{27}-\left(\frac{1}{27}\right)^2+\ldots\right]$$

$$\approx 3[1+.03704-.001372]=3[1.035668]=3.107004$$

Thus $\sqrt{30}\approx 3.107$

Example 5: Find the coefficient of x^n in the expansion of $\frac{1-x}{(1+x)^2}$

Solution: $\frac{1-x}{(1+x)^2}=(1-x)(1+x)^{-2}$

$$=(-x+1)\left[1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\ldots.+\frac{(-2)(-3)\ldots(-2-r+1)}{r!}x^r+\ldots\right]$$

$$=(-x+1)[1+(-1)2x+(-1)^2 3x^2+\ldots+(-1)^r\times(r+1)x^r+\ldots]$$

$$=(-x+1)[1+(-1)2x+(-1)^2 3x^2+\ldots.+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n+\ldots]$$

coefficient of $x^n=(-1)(-1)^{n-1}n+(-1)^n(n+1)$

$$=(-1)^n n+(-1)^n(n+1)$$

$$=(-1)^n[n+(n+1)]$$

$$=(-1)\ .(2\ \ +1)$$

Example 6: If x is so small that its cube and higher power can be neglected, show that

$$\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$$

Solution: $\sqrt{\frac{1-x}{1+x}}=(1-x)^{1/2}(1+x)^{-1/2}$

$$=\left[1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\ldots\right]\left[1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2+\ldots\right]$$

$$=\left[1-\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right]\left[1-\frac{1}{2}x+\frac{3}{8}x^2+\ldots\right]$$

$$=\left[\left(1-\frac{1}{2}x+\frac{3}{8}x^2\right)+\left(-\frac{1}{2}x+\frac{1}{4}x^2\right)-\frac{1}{8}x^2+\ldots\right]$$

$$=1-\left(\frac{1}{2}+\frac{1}{2}\right)x+\left(\frac{3}{8}+\frac{1}{4}-\frac{1}{8}\right)x^2+\ldots$$

$$\approx 1-x+\frac{1}{2}x^2$$

Example 7: If *m* and *n* are nearly equal, show that

$$\left(\frac{5m-2n}{3n}\right)^{1/3}\approx\frac{m}{m+2n}+\frac{n+m}{3n}$$

Solution: Put $m = n + h$ (here h is so small that its square and higher powers can be neglected)

$$\text{L.H.S.} = \left(\frac{5m-2n}{3n}\right)^{1/3} = \left(\frac{5(n+h)-2n}{3n}\right)^{1/3} = \left(\frac{3n+5h}{3n}\right)^{1/3}$$

$$= \left(1+\frac{5h}{3n}\right)^{1/3}$$

$$\approx 1+\frac{5h}{9h} \quad \text{(neglecting square and higher powers of } h) \quad \text{(i)}$$

$$\text{R.H.S.} = \frac{m}{m+2n} + \frac{n+m}{3n}$$

$$= \frac{n+h}{3n+h} + \frac{2n+h}{3n}$$

$$= \frac{(n+h)}{3n}\left(\frac{1}{1+\frac{h}{3n}}\right) + \left(\frac{2}{3}+\frac{h}{3n}\right)$$

$$= (n+h)\frac{1}{3n}\left(1+\frac{h}{3n}\right)^{-1} + \left(\frac{2}{3}+\frac{h}{3n}\right)$$

$$= \left(\frac{1}{3}+\frac{h}{3n}\right)\left(1-\frac{h}{3n}+\dots\right) + \left(\frac{2}{3}+\frac{h}{3n}\right)$$

$$= \left[\frac{1}{3}+\left(-\frac{h}{9n}+\frac{h}{3n}\right)+\dots\right] + \frac{2}{3}+\frac{h}{3n}$$

$$\approx 1+\frac{5h}{9n} \quad \text{(neglecting square and higher powers of } h) \quad \text{(ii)}$$

From (i) and (ii), we have the result.

Example 8: Identify the series: $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\dots$ as a binomial expansion and find its sum

Solution: Let the given series be identical with.

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots \quad \text{(A)}$$

We know that (A) is expansion of $(1+x)^n$ for $|x| < 1$ and n is not a positive integer. Now comparing the given series with (A) we get:

$$nx = \frac{1}{3} \quad \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{3.6} \quad \text{(ii)}$$

From (i), $x = \frac{1}{3n}$

Now substitution of $x = \frac{1}{3n}$ in (ii) gives

$$\frac{n(n-1)}{2!}.\left(\frac{1}{3n}\right)^2 = \frac{1}{6} \text{ or } \frac{n(n-1)}{2!}.\frac{1}{9n^2} = \frac{1}{6}$$

or $\quad n-1 = 3n \Rightarrow n = -\frac{1}{2}$

Putting $n = -\frac{1}{2}$ in (iii), we get

$$x = \frac{1}{3\left(-\frac{1}{2}\right)} = -\frac{2}{3}$$

Thus the given series is the expansion of $\left[1+\left(-\frac{2}{3}\right)\right]^{-1/2}$ or $\left(1-\frac{2}{3}\right)^{-1/2}$

Hence the sum of the given series $=\left(1-\frac{2}{3}\right)^{-1/2}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}=(3)^{1/2}$

$=\sqrt{3}$

Example 9: For $y=\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1.3}{2^2.2!}\left(\frac{4}{9}\right)^2+\frac{1.3.5}{2^3.3!}\left(\frac{4}{9}\right)^3+...$

show that $5y^2 + 10y - 4 = 0$

Solution: $y=\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2+\frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3+...$ (A)

Adding 1 to both sides of (A), we obtain

$1+y=1+\frac{1}{2}\left(\frac{4}{9}\right)+\frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2+\frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3+...$ (B)

Let the series on the right side of (B) be identical with

$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$

which is the expansion of $(1+x)^n$ for $|x|<1$ and n is not a positive integer. On comparing terms of both the series, we get

$nx=\frac{1}{2}.\left(\frac{4}{9}\right)$ (i)

$\frac{n(n-1)}{2!}x^2=\frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2$ (ii)

From (i), $x=\frac{2}{9n}$ (iii)

Substituting $x=\frac{2}{9n}$ in (ii), we get

$\frac{n(n-1)}{2}.\left(\frac{2}{9n}\right)^2=\frac{3}{8}.\frac{16}{81}$ or $\frac{n(n-1)}{2}.\frac{4}{81n^2}=\frac{3}{8}.\frac{16}{81}$

or $2(n-1) = 6n$ or $n-1 = 3n \Rightarrow n=-\frac{1}{2}$

Putting $n=-\frac{1}{2}$ in (iii), we get

$$x=\frac{2}{9\left(-\frac{1}{2}\right)}=-\frac{4}{9}$$

Thus $1+y=\left(1-\frac{4}{9}\right)^{-1/2}=\left(\frac{5}{9}\right)^{-1/2}=\left(\frac{9}{5}\right)^{1/2}$

$=\frac{3}{\sqrt{5}}$

or $\sqrt{5}(1+y)=3$ (iv)

Squaring both the sides of (iv), we get

$5(1 + 2y + y^2) = 9$

or $5y^2 + 10y - 4 = 0$

Exercise 8.3

1. Expand the following upto 4 terms, taking the values of x such that the expansion in each case is valid.

i) $(1-x)^{1/2}$ ii) $(1+2x)^{-1}$ iii) $(1+x)^{-1/3}$ iv) $(4-3x)^{1/2}$

v) $(8-2x)^{-1}$ vi) $(2-3x)^{-2}$ vii) $\frac{(1-x)^{-1}}{(1+x)^2}$ viii) $\frac{\sqrt{1+2x}}{1-x}$

ix) $\frac{(4+2x)^{1/2}}{2-x}$ x) $(1+x-2x^2)^{\frac{1}{2}}$ xi) $\left(1-2x+3x^2\right)^{\frac{1}{2}}$

2. Using Binomial theorem find the value of the following to three places of decimals.

i) $\sqrt{99}$ ii) $(.98)^{\frac{1}{2}}$ iii) $(1.03)^{\frac{1}{3}}$ iv) $\sqrt[3]{65}$

v) $\sqrt[4]{17}$ vi) $\sqrt[5]{31}$ vii) $\dfrac{1}{\sqrt[3]{998}}$ viii) $\dfrac{1}{\sqrt[5]{252}}$

ix) $\dfrac{\sqrt{7}}{\sqrt{8}}$ x) $(.998)^{-\frac{1}{3}}$ xi) $\dfrac{1}{\sqrt[6]{486}}$ xii) $(1280)^{\frac{1}{4}}$

3. Find the coefficient of x^n in the expansion of

i) $\dfrac{1+x^2}{(1+x)^2}$ ii) $\dfrac{(1+x)^2}{(1-x)^2}$ iii) $\dfrac{(1+x)^3}{(1-x)^2}$

iv) $\dfrac{(1+x)^2}{(1-x)^3}$ v) $(1-x+x^2-x^3+\ldots)^2$

4. If x is so small that its square and higher powers can be neglected, then show that

i) $\dfrac{1-x}{\sqrt{1+x}} \approx 1-\dfrac{3}{2}x$ ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\dfrac{3}{2}x$

iii) $\dfrac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x} \simeq \dfrac{1}{4}-\dfrac{17}{384}x$

iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2+\dfrac{25}{4}x$

v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx 4\left(1-\dfrac{5x}{6}\right)$

vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

vii) $\dfrac{\sqrt{4-x}+(8-x)^{1/3}}{(8-x)^{1/3}} \approx 2-\dfrac{1}{12}x$

5. If x is so small that its cube and higher power can be neglected, then show that

i) $\sqrt{1-x-2x^2} \approx 1-\dfrac{1}{2}x-\dfrac{9}{8}x^2$ ii) $\sqrt{\dfrac{1+x}{1-x}} \approx 1+x+\dfrac{1}{2}x^2$

6. If x is very nearly equal 1, then prove that $px^p - qx^q \approx (p-q)x^{p+q}$

7. If $p-q$ is small when compared with p or q, show that

$$\frac{(2n+1)p+(2n-1))q}{(2n-1)p+(2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$

8. Show that $\left[\dfrac{n}{2(n+N)}\right]^{1/2} \approx \dfrac{8n}{9n-N}-\dfrac{n+N}{4n}$ where n and N are nearly equal.

9. Identify the following series as binomial expansion and find the sum in each case.

i) $1-\dfrac{1}{2}\left(\dfrac{1}{4}\right)+\dfrac{1.3}{2!4}\left(\dfrac{1}{4}\right)^2-\dfrac{1.3.5}{3!8}\left(\dfrac{1}{4}\right)^3+\ldots$

ii) $1-\dfrac{1}{2}\left(\dfrac{1}{2}\right)+\dfrac{1.3}{2.4}\left(\dfrac{1}{2}\right)^2-\dfrac{1.3.5}{2.4.6}\left(\dfrac{1}{2}\right)^3+\ldots$

iii) $1+\dfrac{3}{4}+\dfrac{3.5}{4.8}+\dfrac{3.5.7}{4.8.12}+\ldots$

iv) $1-\dfrac{1}{2}.\dfrac{1}{3}+\dfrac{1.3}{2.4}\left(\dfrac{1}{3}\right)^2-\dfrac{1.3.5}{2.4.6}\left(\dfrac{1}{3}\right)^3+\ldots$

10. Use binomial theorem to show that $1+\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.12}+\ldots=\sqrt{2}$

11. If $y=\dfrac{1}{3}+\dfrac{1.3}{2!}\left(\dfrac{1}{3}\right)^2+\dfrac{1.3.5}{3!}\left(\dfrac{1}{3}\right)^3+\ldots$, then prove that $y^2+2y-2-0$

12. If $2y=\dfrac{1}{2^2}+\dfrac{1.3}{2!}.\dfrac{1}{2^4}+\dfrac{1.3.5}{3!}.\dfrac{1}{2^6}+\ldots$ then prove that $4y^2+4y-1=0$

13. If $y=\dfrac{2}{5}+\dfrac{1.3}{2!}.\left(\dfrac{2}{5}\right)^2+\dfrac{1.3.5}{3!}\left(\dfrac{2}{5}\right)^3+\ldots$ then prove that $y^2+2y-4=0$